Special Holonomy Manifolds in String Theory & M-Theory Reductions

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April 2023

This piece of work is a result of my own work except where it forms an assessment based on group project work. In the case of a group project, the work has been prepared in collaboration with other members of the group. Material from the work of others not involved in the project has been acknowledged and quotations and paraphrases suitably indicated.

Abstract

Kaluza-Klein theory proposed a unification of gravity and electromagnetism by dimensionally reducing the Einstein-Hilbert action with a fifth circular dimension. We take inspiration from this and consider theories in ten and eleven dimensions, specifically Type IIA string theory and M-theory, where we obtain effective four-dimensional theories by dimensionally reducing the higher dimensional theories on Calabi-Yau- and G_2-manifolds. We give examples of constructions of both Calabi-Yau and G_2-manifolds, particularly orbifold constructions. We find that the resulting four-dimensional theory from M-theory reductions contains the standard model gauge group within the resulting gauge symmetry, as well as predicting a number of gauge fields and scalar fields.

My gratitude to my supervisors, AB & AD, for their guidance, to CK for his collaboration and friendship, and to JG, SG, EG, JF for their unwavering support.

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1 Introduction

In this project I shall be discussing Kaluza-Klein theory and it's modern day developments in string theory and M-theory. The theory concerns adding additional dimensions of space to an existing theory of, say, gravity, and then 'dimensionally reducing' these extra dimensions to obtain a lower dimensional theory that models our universe. As we will see, simpler higher-dimensional theories can provide unified theories of physics in lower dimensions. The reason for considering string theory and M-theory reductions later on in this thesis is because they provide us with a more general unified theory than the original Kaluza-Klein theory. A specific focus of this thesis is how topological and geometrical information of the extra dimensions influences the resulting 4D theory after dimensional reduction.

I will be assuming that the reader is comfortable with differential geometry, the basics of topology, and Riemannian geometry, as well as general relativity, Lagrangian mechanics, and classical field theory. Knowledge of group theory is also assumed for more common Lie groups such as $O(n), U(n)$.

In this section we shall take a look at the original Kaluza-Klein theory in Subsection 1.1, and then in Subsection 1.2 we will introduce the required topological and geometric concepts for dealing with higher dimensional theories.

1.1 Kaluza-Klein Theory

After Einstein published his theory of general relativity in 1915, Kaluza suggested to Einstein in 1919 the idea of adding an additional dimensional to the Einstein-Hilbert action to achieve a unification of gravity and electromagnetism. In 1926, Klein then suggested that the additional dimension should be S^1, which can result in quantized theories [15]. Almost all of the theories that we will be considering have the Einstein-Hilbert term in their action, so it will be good for us to consider dimensional reductions of this action to begin with.

Definition 1.1 (Einstein-Hilbert Action)**.** *We define the Einstein-Hilbert action, which results in the Einstein Field Equations, as*

$$S = \int_M d^n x \sqrt{-g}\mathscr{R} \tag{1}$$

where $g = det(g_{IJ})$ and $\mathscr{R}$ is the Ricci Scalar of the metric g [6]. Here, M is just some n-dimensional manifold that we have not yet specified.

Notice that the dimension of the integral in Definition 1.1 is general, as well as the manifold we're integrating over. So, we can follow the idea of Kaluza & Klein, by going from a 5D Einstein-Hilbert action with $M = \mathbb{M}_4 \times S^1$ and make the Kaluza-Klein Ansatz [1] [20]:

[1]There are different versions of this Ansatz, most of which give very similar results.

Definition 1.2 (Kaluza-Klein Ansatz)**.** *Let $\tilde{g}_{IJ}$ be a 5-dimensional metric over $M = \mathbb{M}_4 \times S^1$. Then we make the following ansatz, known as the Kaluza-Klein Ansatz:*

$$\tilde{g}_{IJ} = \phi^{-\frac{1}{3}} \begin{pmatrix} g_{\mu\nu} + \phi A_\mu A_\nu & \phi A_\mu \\ \phi A_\nu & \phi \end{pmatrix} \tag{2}$$

where the I,J indices run from 0 to 4, referring to the whole space $M = \mathbb{M}_4 \times S^1$, where μ and ν run from 0 to 3, over just the Minkowski space. In all of our examples, the indices I,J will run over the whole manifold while Greek indices run over just our Minkowski space. In this Ansatz we have taken a Fourier expansion of the fields over the S^1 coordinate, and have chosen the zero modes to be the only contributing factor, i.e.

$$\tilde{g}_{IJ}(x,y) = \sum_n \tilde{g}_{IJn}(x) e^{iny/R} \tag{3}$$

but we omit all non-zero n. Instead of referring to the zero modes as $g_{\mu\nu 0}, A^0_\mu, \phi_0$, we shall simply omit the 0-index. Here x refers to the coordinates on the Minkowski space, y is the coordinate on S^1, and R is the radius of the S^1 [20].

Note that this ansatz does not result in a loss of generality, as it's simply a choice of parametrisation of the metric. Any object with a Tilde refers to an object in 5D, while the objects without a Tilde refer to 4D objects - we will follow this convention of tildes representing higher-dimensional objects throughout this thesis.

Lemma 1.1. *Let our metric from Definition 1.2 have determinant $\tilde{g} = det\tilde{g}_{IJ}$. We can then write it in terms of only 4D objects*

$$\tilde{g} = det g_{IJ} \tag{4}$$

$$= (\phi^{-\frac{1}{3}})^5 (\phi det(g_{\mu\nu}) + \phi^2 det(A_\mu A_\nu) - \phi^2 det(A_\mu A_\nu)) \tag{5}$$

$$= \phi^{-\frac{2}{3}} g \tag{6}$$

where $g = det g_{\mu\nu}$ is the determinant of the 4D metric on $\mathbb{M}_4$.

We are now in a position to begin the dimensional reduction of the Einstein-Hilbert action. In the following theorem we will assume the 5D action, but it is not too difficult to generalise this.

Theorem 1.1 (Dimensional Reduction of 5D Einstein-Hilbert action)**.** *We have already used the Kaluza-Klein Ansatz from Definition 1.2 to find the metric and it's determinant in terms of 4D fields. Now we want to take our 5D $\mathscr{R}$ and write it in terms of 4D fields also. Taking our expansion of the Ricci scalar from Exercise 7.3 of [12], and putting a tilde on 5D quantities:*

$$\tilde{\mathscr{R}} = \mathscr{R} - 2e^{-\sigma} \nabla^2 e^{\sigma} - \frac{1}{4} e^{2\sigma} F_{\mu\nu} F^{\mu\nu} \tag{7}$$

where $\phi = e^{2\sigma}$, $\sigma = \frac{1}{2}\log\phi$. *Evaluating the second term of this gives us*

$$\nabla^2 e^\sigma = \partial_\mu \partial^\mu e^\sigma \tag{8}$$

$$= (\partial_\mu \partial^\mu \sigma)e^\sigma + (\partial_\mu \sigma \partial^\mu \sigma)e^\sigma \tag{9}$$

$$\Rightarrow -2e^{-\sigma}\nabla^2 e^\sigma = -2(\partial_\mu \partial^\mu \sigma) - 2(\partial_\mu \sigma \partial^\mu \sigma) \tag{10}$$

$$= -(\partial_\mu \partial^\mu \log\phi) - \frac{1}{2}(\partial_\mu \log\phi \partial^\mu \log\phi) \tag{11}$$

$$= \frac{-2\phi\partial_\mu \partial^\mu \phi + \partial_\mu \phi \partial^\mu \phi}{2\phi^2} \tag{12}$$

but as we choose to discard total derivatives in the Lagrangian, we rewrite the second term as a total derivative and end up getting the term in [12] [2]*. Substituting all of this into the 5D Einstein-Hilbert action gives us*

$$S = 2\pi R \int_{\mathbb{M}_4} d^4x\sqrt{-g}[\mathscr{R} - \frac{1}{4}\phi F_{\mu\nu}F^{\mu\nu} + \frac{3}{2}\frac{\partial_\mu \phi \partial^\mu \phi}{\phi^2}] \tag{13}$$

Notice here that we now have an Einstein-Hilbert term, a Maxwell term and a scalar field all in 4D. This then looks like a theory of gravity and electromagnetism in 4D from just a 5D theory of gravity.

So beginning with a 5D theory of pure gravity, we have obtained unification of gravity and electromagnetism, as well as obtaining a massless scalar field. This is sometimes referred to as the 'Kaluza-Klein miracle', and lays the foundation of this thesis. This idea by Kaluza & Klein to unify two separate forces by suggesting that in higher dimensions they are one in the same is the motivation behind string theory, and this is where we are headed in this project. If we can unify two seemingly different forces with just the introduction of a single extra dimension, what will happen when we add even more?

It is very important for us to note the resulting gauge symmetry of our 4D theories - for example, a theory of a vector field would only be considered as electromagnetism if it had the corresponding $U(1)$ gauge symmetry. We shall show that the vector fields A_μ from the Kaluza-Klein ansatz have a U(1) gauge symmetry in the following:

Theorem 1.2 ($U(1)$ Gauge Symmetry of 4D Kaluza-Klein Reduction)**.** *The fields* A_μ *with field strength* $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ *in Equation (13) have a* $U(1)$ *gauge symmetry.*

Proof: If we let A_μ *transform like a gauge field under a* $U(1)$ *symmetry,*

$$A_\mu \to A_\mu + \partial_\mu \alpha \tag{14}$$

then we get:

$$F_{\mu\nu}F^{\mu\nu} \to (\partial_\mu(A_\nu + \partial_\nu \alpha) - \partial_\nu(A_\mu + \partial_\mu \alpha))(\partial^\mu(A^\nu + \partial^\nu \alpha) - \partial^\nu(A^\mu + \partial^\mu \alpha)) \tag{15}$$

$$= (F_{\mu\nu} + \partial_\mu \partial_\nu \alpha - \partial_\nu \partial_\mu \alpha)(F^{\mu\nu} + \partial^\mu \partial^\nu \alpha - \partial^\nu \partial^\mu \alpha) \tag{16}$$

$$= F_{\mu\nu}F^{\mu\nu} \tag{17}$$

by symmetry of the partial derivatives acting on α. □

[2] We end up with a different coefficient to [12]

If we were to achieve a reduction that looked like the standard model, then we would hope that the corresponding gauge symmetry contains $SU(3) \times SU(2) \times U(1)$ [2]. This will be an important idea to note much later on in this thesis.

Something else that will be of importance to us is the 'consistency' of our reductions. Reducing a higher dimensional action to a lower dimension does not mean that we can fully trust the latter:

Theorem 1.3 (Consistent Reduction)**.** *Let S_{n+k} be an action in (n+k)-dimensions, and S_n an action dimensionally reduced from S_{n+k}. If the equations of motion derived from S_n satisfy the equations of motion derived from S_{n+k}, then we call S_n a consistent reduction [18].*

This theorem essentially means that for us to trust that any ansatz we make in our reduction is reliable, we require that the equations of motion in the lower-dimensional theory are essentially 'the same' as the higher dimensional equations of motion. We have from [18] that the reduction in Theorem 1.1 is consistent, and so we are able to treat this as a realistic physical theory.

Some additional examples of Kaluza-Klein reductions on S^1 can be found in Appendix B. In this appendix we consider some interesting results that we get in 4D from a 5D reduction without checking that these reductions are 'consistent'. So while we get some intriguing theories, we could never consider them as reliable physical theories.

In this introductory Subsection we have seen that a possible way to achieve a unified theory of physics is by considering our fields in 4D as a unified field in higher-dimensional theories. Already we have seen a unification of gravity and electromagnetism by considering just one extra spatial dimension. This motivates us to consider string theory and M-theory, popular contenders of a unified theory, which exist in 10- and 11-dimensional space. In the following chapters we will introduce the more advanced 6D geometry that is used in 10D string theories and how reductions work on them, and then we'll introduce the 7D geometry used in 11D M-theory, and consider the reduction of this theory. We hope that we shall see the resulting 4D theory contains greater unification than the Kaluza-Klein theory.

1.2 Forms and Cohomology

To consider theories in higher dimensions, we will discuss some essential tools - exterior calculus of differential forms, and cohomology. These will make it far easier for us to talk about the geometry of these higher dimensions.

The first mathematical object we wish to define is that of a Differential Form. The reason for this is that we will end up writing some very extensive equations later on, and a differential form allows us to simplify many expressions. In the same way that the Einstein Summation Convention allows us to implicitly sum over indices and reference entire tensors by assigning indices to individual elements, a differential form allows us to go even further with condensing our notation. They also end up being incredibly important objects in their own right when dealing with concepts such as cohomology, which we cover in this Subsection.

Definition 1.3 (Differential Form)**.** *A differential p-form is defined as totally antisymmetric tensor of rank (0,p) [14].*

This might seem like a very basic notion, but they end up having a very powerful physical interpretation. Before we give an example of a differential form, it's worth discussing notation for tangent and cotangent spaces of manifolds. If we have a point $q \in M$, where M is a manifold, then we write the tangent space at q as T_qM, and the cotangent space as T_q^*M. The bases of these spaces are given by $\{\frac{\partial}{\partial x^k}\}_{k\in I}$ and $\{dx^k\}_{k\in I}$ respectively, for index set $I = \{1,...,n\}$ for n-dimensional M. Naturally, as T_q^*M is a dual space to T_qM, the inner product between the two is given by

$$< dx^k, \frac{\partial}{\partial x^l} >= \delta_l^k \tag{18}$$

where we have borrowed the notation from [5]. However, for almost all of our discussions we will rely on elements in T_p^*M, i.e. the cotangent vectors.

Let's now consider various dimensions of p-form. If we let $p = 0$, and take the definition of a 0-form from Definition 1.3, then we simply end up with an antisymmetric tensor of rank (0,0), i.e. a scalar. What then do we mean by a 1-form? Well, this would be a tensor of rank (0,1), which we could write as T^k, i.e. a vector. Similarly, a 2-form can be written as $T^{kl} = -T^{lk}$.

Generally, we can pick the basis $\{dx^k\}_{k\in I}$ for our 1-forms, and for p-forms we have the space of (smooth) p-forms at point $q \in M$ defined as [14]

$$\Omega_q^p(M) = span\{dx^{k_1} \wedge ... \wedge dx^{k_p}\}, k_1 < ... < k_p \tag{19}$$

where we define the wedge product as

$$dx^{k_1} \wedge ... \wedge dx^{k_p} = \frac{1}{p!}\varepsilon_{k_1...k_p} dx^{k_1} \otimes ... \otimes dx^{k_p} \tag{20}$$

which is essentially the sum of the even permutations of the tensor products of the basis elements minus the sum of the odd permutations (note the Einstein Summation Convention being used here) [14].

If M is an n-dimensional manifold, then

$$dim\Omega_q^p(M) = \frac{n!}{p!(n-p)!} \tag{21}$$

which we have taken from [14].

Definition 1.4 (Differential Form p-form in coordinates)**.** *A general p-form $F \in \Omega_q^p(M)$ can be written as*

$$F = F_{k_1..k_p} dx^{k_1} \wedge ... \wedge dx^{k_p} \tag{22}$$

where $F_{k_1..k_p}$ is anti-symmetric [5]. Some authors use a normalization constant of $\frac{1}{p!}$, but we will not include these in our definitions.

Definition 1.5. *Let $F \in \Omega^p(M), G \in \Omega^q(M)$, then we have*

$$F \wedge G = F_{k_1..k_p} G_{l_1..l_q} dx^{k_1} \wedge ... \wedge dx^{k_p} \wedge dx^{l_1} \wedge ... \wedge dx^{l_q} = (-1)^{pq} G \wedge F \tag{23}$$

such that $F \wedge G \in \Omega^{p+q}(M)$ [5].

Now that we've gone over some essentials we can introduce an incredibly useful tool for differential forms:

Definition 1.6 (Exterior Derivative). *Define $d : \Omega^p_q(M) \to \Omega^{p+1}_q(M)$ as the exterior derivative of a p-form, such that*

$$dF = \frac{\partial F_{k_1...k_p}}{\partial x^l} dx^l \wedge dx^{k_1} \wedge ... \wedge dx^{k_p} \tag{24}$$

which is obviously a (p+1)-form [14].

We can introduce an incredibly important and fundamental property of the exterior derivative which, while simple, is core to the study of cohomology later on:

Lemma 1.2. *If we take d^2F by applying the exterior derivative twice on a p-form then we get [5]*

$$d^2 = 0 \tag{25}$$

Proof:

$$\begin{aligned}
d^2F &= \frac{\partial^2 F_{k_1..k_p}}{\partial x^m \partial x^l} dx^m \wedge dx^l \wedge dx^{k_1} \wedge ... \wedge dx^{k_p} && (26)\\
&= -\frac{\partial^2 F_{k_1..k_p}}{\partial x^m \partial x^l} dx^l \wedge dx^m \wedge dx^{k_1} \wedge ... \wedge dx^{k_p} && (27)\\
&= -\frac{\partial^2 F_{k_1..k_p}}{\partial x^l \partial x^m} dx^l \wedge dx^m \wedge dx^{k_1} \wedge ... \wedge dx^{k_p} && (28)\\
&= -\frac{\partial^2 F_{k_1..k_p}}{\partial x^m \partial x^l} dx^m \wedge dx^l \wedge dx^{k_1} \wedge ... \wedge dx^{k_p} && (29)\\
&= -d^2F && (30)\\
&\Rightarrow d^2F = 0 && (31)
\end{aligned}$$

so we see that for any differential form we have $d^2F = 0$, and therefore $d^2 = 0$. □

Lemma 1.3. *Let $F_{k_1...k_p}$ be a symmetric tensor, then the form*

$$F = F_{k_1...k_p} dx^{k_1} \wedge ... \wedge dx^{k_p} = 0 \tag{32}$$

Proof:

$$\begin{aligned}
F &= F_{k_1 k_2...k_p} dx^{k_1} \wedge dx^{k_2} ... \wedge dx^{k_p} && (33)\\
&= F_{k_2 k_1...k_p} dx^{k_1} \wedge dx^{k_2} ... \wedge dx^{k_p} && (34)\\
&= -F_{k_2 k_1...k_p} dx^{k_2} \wedge dx^{k_1} ... \wedge dx^{k_p} && (35)\\
&= -F && (36)\\
&\Rightarrow F = 0 \quad \square && (37)
\end{aligned}$$

Definition 1.7 (Derivative of Wedge Products)**.** *Let F be a p-form, G be a q-form. Then [4]*

$$d(F \wedge G) = dF \wedge G + (-1)^p F \wedge dG \tag{38}$$

Definition 1.8 (Exact Forms)**.** *If a form can be written as*

$$F_p = dG_{p-1} \tag{39}$$

then we say it is exact [5].

Definition 1.9 (Closed Forms)**.** *If a form has that*

$$dF = 0 \tag{40}$$

then we say it is closed [5].

Lemma 1.4. *If we have an exact form F, then it can be written as*

$$F = dG \Rightarrow dF = d^2G = 0 \tag{41}$$

which means that every exact form is closed, by Lemma 1.2.

Now feels like an appropriate time to see some examples of exterior calculus and differential forms. One that is quite useful is to simply apply exterior derivatives to 0-, 1-, and 2-forms in Euclidean space $\mathbb{R}^3$, as well as looking at the Electromagnetic Tensor as a differential form:

Example 1.1 (Vector Calculus results via exterior calculus)**.** *We will follow a commonly used example, from both [5], [14]. However, we will add an interesting extension towards the end. Denote V_p as a p-form for $p \in \{0,1,2\}$. Then we can write:*

$$V_0 = f(x,y,z) \tag{42}$$

$$V_1 = f_x(x,y,z)dx + f_y(x,y,z)dy + f_z(x,y,z)dz \tag{43}$$

$$V_2 = f_{xy}dx \wedge dy + f_{yz}dy \wedge dz + f_{zx}dz \wedge dx \tag{44}$$

where f are 0-, 1-, or 2-dimensional maps in $\mathbb{R}^3$, depending on the value of p. Taking the exterior derivatives of each of these

$$dV_0 = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \tag{45}$$

$$dV_1 = (\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y})dx \wedge dy + (\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z})dy \wedge dz + (\frac{\partial f_z}{\partial x} - \frac{\partial f_x}{\partial z})dz \wedge dx \tag{46}$$

$$dV_2 = (\frac{\partial f_{yz}}{\partial x} + \frac{\partial f_{zx}}{\partial y} + \frac{\partial f_{xy}}{\partial z})dx \wedge dy \wedge dz \tag{47}$$

we see each application of exterior derivative corresponds to the vector calculus operators $\nabla V_0, \nabla \times V_1, \nabla \cdot V_2$, depending on the dimension of V_p. This is example is displayed both in [14] and [5]. However, what these don't show is how Lemma 1.2 gives us some results from multivariate calculus.

For example, we have seen that $dV_0 \equiv \nabla V_0$ is a 1-form. A standard result from multivariate calculus that readers should be familiar with is that $\nabla \times \nabla V_0 = 0$. Well, considering that $dV_0 \in \Omega^1(\mathbb{R}^3)$, and that $dV_1 \equiv \nabla \times V_1$, we then have that $d(dV_0) = d^2V_0 = 0 \Rightarrow \nabla \times (\nabla V_0) = 0$. Similarly, we get $\nabla \cdot (\nabla \times V_1) = 0$.

Example 1.2 (Electromagnetic Tensor)**.** *We begin by defining the Electromagnetic four-potential simply as a 1-form $A = A_\nu dx^\nu$. We then take the exterior derivative of this to get*

$$dA = \frac{\partial A_\nu}{\partial x^\mu} dx^\mu \wedge dx^\nu \tag{48}$$

$$= \partial_\mu A_\nu dx^\mu \wedge dx^\nu \tag{49}$$

and now we can recognise that as $\mu, \nu \in \{0,..,3\}$, we can fix $\mu < \nu, \mu \neq \nu$ and split this into

$$dA = \partial_\mu A_\nu dx^\mu \wedge dx^\nu + \partial_\mu A_\nu dx^\nu \wedge dx^\mu \tag{50}$$

$$= \partial_\mu A_\nu dx^\mu \wedge dx^\nu - \partial_\nu A_\mu dx^\mu \wedge dx^\nu \tag{51}$$

$$= (\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \wedge dx^\nu \tag{52}$$

$$\Rightarrow F = dA = F_{\mu\nu} dx^\mu \wedge dx^\nu \tag{53}$$

where $F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)$ as we would expect for the Electromagnetic Tensor. We know by $d^2 = 0$ that $dF = 0$, which gives us one of Maxwell's Equations.

Now that we've seen some basic examples, we would like to extend our understanding of the relationship between forms of different dimensions. So far we've seen the relationship between a p-form and a (p+1)-form is through the exterior derivative, but a more extensive relationship can be found through the Hodge Star.

Definition 1.10 (Hodge Star Operator)**.** *Let $F \in \Omega_q^p(M)$ be a p-form at point $q \in M$ for smooth Riemannian manifold (M,g) of dimension n. Then we can define the Hodge Star Operator as $\star : \Omega_q^p(M) \to \Omega_q^{(n-p)}(M)$ such that [14]*

$$\star F = g^{1/2} F_{k_1 \ldots k_p} \varepsilon^{k_1 \ldots k_p}_{l_{p+1} \ldots l_n} dx^{l_{p+1}} \wedge \ldots \wedge dx^{l_n} \tag{54}$$

where $g^{1/2}$ is the root of the determinant of the metric of M. As we can see, this is an (n-p)-form, thus giving us a relationship between the two dimensions of forms.

Lemma 1.5. *We can apply the Hodge star twice to a p-form to arrive back in the space $\Omega_q^p(M)$, with the result for Riemannian (M,g) being [14]*

$$\star\star F = (-1)^{p(n-p)} F \tag{55}$$

The proof of this Lemma 1.5 can be found in [14], but we won't include it here as it's not particularly relevant, as well as being quite index-heavy.

Definition 1.11 (Exterior Product of forms)**.** *The exterior product of two p-forms is given by*

$$F \wedge \star G = g^{\frac{1}{2}} F_{k_1 \ldots k_p} G^{k_1 \ldots k_p} dx^1 \wedge \ldots \wedge dx^n \tag{56}$$

which is evidently an n-form and symmetric [14].

Definition 1.12 (Inner Product of forms)**.** *From the exterior product we can define the inner product on the space* $\Omega_q^p(M)$ *as [5]*

$$< F, G > = \int F \wedge \star G \tag{57}$$

$$= \int_M g^{\frac{1}{2}} F_{k_1 \ldots k_p} G^{k_1 \ldots k_p} dx^1 \ldots dx^n \tag{58}$$

Notice here that when we write our dx^k *terms in an integral, we actually mean this to be wedge products, but these are usually omitted. Note that this is also symmetric.*

Definition 1.13 (Intersection Numbers)**.** *We can define a topological invariant of an n-dimensional manifold M, called the intersection numbers:*

$$\kappa_{i_1 .. i_k} = \int_M \omega_{i_1} \wedge .. \wedge \omega_{i_k} \tag{59}$$

where $\omega_{i_1}, \ldots, \omega_{i_k}$ *are a basis of p-forms such that* $k \times p = n$*. We have examples of the intersection numbers of just two forms given in [5] and for a triplet of 2-forms on a 6-dimensional manifold in [2]. Here we give a general definition. We will also allow a combination of forms of different dimension to define intersection numbers.*

Theorem 1.4 (Stokes' Theorem for Forms)**.** *Given an n-dimensional manifold M with (n-1)-dimensional boundary* ∂M*, and an (n-1)-form A on M, then Stokes' Theorem states [5]*

$$\int_M dA = \int_{\partial M} A \tag{60}$$

Note that we can have the same for a p-dimensional submanifold of M and a (p-1)-form on M.

For more discussion about boundaries of manifolds, see Appendix A. Here we also give physical meaning to differential forms.

Corollary 1.1 (Integration on manifolds without boundary)**.** *Let M be an n-dimensional manifold with no boundary, and let A be an (n-1)-form. Then from Theorem 1.4 we have*

$$\int_M dA = 0 \tag{61}$$

This is a way of saying that total derivatives on forms without boundary vanish.

We are nearly at a point where we can move onto discussing cohomology, but first we need to introduce the concept of the Laplacian operator:

Definition 1.14 (Adjoint Exterior Derivative)**.** *Let the Adjoint Exterior Derivative $d^\dagger$ be given by $d^\dagger : \Omega^p_q(M) \to \Omega^{p-1}_q(M)$ for n-dimensional M such that [14]*

$$d^\dagger F = (-1)^{np+n+1} \star d \star F \tag{62}$$

where it can be shown that $(d^\dagger)^2 = 0$ and $< F_p, dG_{p-1} >=< d^\dagger F_p, G_{p-1} >$. Similarly to the definition of a closed form, we have that a form is co-closed if $d^\dagger F = 0$ [5].

Definition 1.15 (Laplacian Operator)**.** *We define the Laplacian, or the de Rham Operator, as $\triangle : \Omega^p_q(M) \to \Omega^p_q(M)$ such that [5]*

$$\triangle = dd^\dagger + d^\dagger d \tag{63}$$

Definition 1.16 (Harmonic Forms)**.** *We call a p-form F a Harmonic Form if*

$$\triangle F = 0 \tag{64}$$

where F is Harmonic $\iff$ $dF = 0$ and $d^\dagger F = 0$ (i.e. F is both closed and co-closed) [5].

Something we'll need for Chapters 3 and 4 is the action of the exterior derivative and Hodge star on product manifolds. This will be very important for us in Sections 3 and 4, where we will be considering product manifolds extensively via the product of our 4D Minkowski space and the manifolds we have representing our extra dimensions.

Lemma 1.6 (Exterior Derivative on Product Manifolds)**.** *Let $M = M_1 \times M_2$, with exterior derivative d. The manifolds M_1, M_2 each have their own exterior derivative d_1, d_2. Let F_1, F_2 be forms on M_1, M_2 respectively, where F_1 is a p-form. Then using Definition 1.7 we get*

$$d(F_1 \wedge F_2) = (d_1 F_1) \wedge F_2 + (-1)^p F_1 \wedge d_2 F_2 \tag{65}$$

i.e. the exterior derivative on the product manifold acts linearly on each manifold. This is because if F_1 depends on only coordinates on M_1, then the action of d_2 on F_1 vanishes, and the same for F_2.

Lemma 1.7 (Hodge Star on Product Manifolds)**.** *Let us have the same product manifold and forms as Lemma 1.6. Then the Hodge star on the product manifold acts as [13]*

$$\star(F_1 \wedge F_2) = (-1)^{p(n-p)} \star_1 (F_1) \wedge \star_2 (F_2) \tag{66}$$

where F_1 is a p-form and F_2 is a (n-p)-form, $dim(F_1 \wedge F_2) = n$. Here, $\star_1, \star_2$ are the Hodge star operators on M_1, M_2 respectively.

Lemma 1.8. *We can write the Einstein-Hilbert action in the form*

$$S = \int_M \mathscr{R} \star 1 \tag{67}$$

Proof: *Remembering Definition 1.10, and writing the trivial 0-form 1 as some constant function $A = 1$, we get that*

$$\star 1 = \star A = \sqrt{-g} A \varepsilon_{l_1 \dots l_n} dx^{l_1} \wedge \dots \wedge dx^{l_n} \tag{68}$$

$$= n! \sqrt{-g} dx^1 \wedge \dots \wedge dx^n \tag{69}$$

$$\sim \sqrt{-g} d^n x \tag{70}$$

So from this result we can substitute Equation (70) into Equation (67) to recover the form of the action in Definition 1.1. □

For the rest of this thesis we will write the Einstein-Hilbert action like in Lemma 1.8.

Now that we've defined the Laplacian and Harmonic Forms, we can move onto the study of cohomology, which helps us set up important topological properties of our manifolds. Cohomology concerns itself with the number of non-trivial harmonic differential forms on a manifold [3].

Definition 1.17 (de Rham Cohomology)**.** *Let $\mathbb{C}^p = \{F^p | dF^p = 0\}$ be the set of closed p-forms on M, and $\mathbb{B}^p = \{G^p | G^p = dA^{p-1}\}$ be the set of exact p-forms such that for $G^p \in \mathbb{B}^p$ we have $dG^p = d^2 A^{p-1} = 0$. Then we define the de Rham Cohomology of M to be [5]*

$$\mathbb{H}^p = \mathbb{C}^p / \mathbb{B}^p \tag{71}$$

which is again just the quotient set of closed forms and trivially closed (i.e. exact) forms. For two forms in $\mathbb{H}^p$ that differ by an exact form we identify them: [5]

$$F^p \sim F^p + dA^{p-1} \tag{72}$$

Definition 1.18 (Betti Numbers)**.** *Let manifold M have cohomology $\mathbb{H}^p$. Then we define the Betti numbers as*

$$b_p = dim \mathbb{H}^p \tag{73}$$

which is a topological invariant of M [5].

Theorem 1.5 (Euler Characteristic)**.** *Given a manifold M with Betti numbers b_p, we can write it's Euler Characteristic as [5]*

$$\chi(M) = \sum_{p=0}^{n} (-1)^p b_p \tag{74}$$

Definition 1.19 (Künneth Formula)**.** *Let $M = M_1 \times M_2$ be a product manifold. Then the Künneth Formula calculates the Betti numbers of M from the Betti numbers of $M_1 M_2$: [5]*

$$b_k(M) = \sum_{p+q=k} b_p(M_1) b_q(M_2) \tag{75}$$

[3] Rather, de Rham Cohomology concerns itself with differential forms - there are many types of cohomology that consider different objects which are not necessarily equivalent to de Rham cohomology.

Theorem 1.6 (Harmonic Forms in Cohomology)**.** *A cohomology class (i.e. set of closed forms in a cohomology group that differ by an exact form) contains exactly one Harmonic form [5]. This then means that the cohomology group is spanned by the set of harmonic forms of each cohomology class in the group. Therefore the Betti number $b_p = dim H^p(M)$ counts the number of harmonic forms in the cohomology group [5]. The proof of this can be found in [5], but it's not particularly important for the rest of this thesis so we won't include it here.*

2 Calabi-Yau Manifolds

In Subsection 1.1 we saw that taking a higher-dimensional theory and dimensionally reducing the extra dimensions beyond the four-dimensional universe that we experience gives us a richer theory than we began with. So far we've only given our space a single extra dimension S^1, but we'd like to see how much further we can go.

Berger's List tells us about what the holonomy of particular manifolds in a given dimension must be. We are interested in the 6- and 7-dimensional cases, so we will discuss these. The list classifies simply-connected manifolds (that are not locally a product manifold or locally symmetric) by their holonomy group hol(g). The original list that Berger gave was incomplete, so from [10] we have that:

- For 2n-dimensional (M,g) with $hol(g) \subseteq U(n)$, (M,g) is Kähler
- For 2n-dimensional (M,g) with $hol(g) \subseteq SU(n)$, (M,g) is Calabi-Yau
- For 7-dimensional (M,g) with $hol(g) \subseteq G_2$ we call M a G_2-manifold

In the former cases we choose n=3 to get 6-dimensional Kähler and Calabi-Yau manifolds. Clearly all Calabi-Yau manifolds are also Kähler, and from [10] we have that a Kähler manifold is Calabi-Yau iff it is also Ricci-flat. This list is important for us as the extra 6 dimensions of string theory are thus required to be Calabi-Yau and the extra 7 dimensions of M-theory are required to be G_2-manifolds. Manifolds in Berger's List are sometimes referred to as Special Holonomy Manifolds, and they play a key role in this thesis.

It's a well known fact even outside of physics literature that string theory predicts that the universe has an additional 6 or 7 dimensions that are so small that we cannot detect them. This could be exciting to us as we're looking at unification from extra dimensions, and string theory is thought to be able to combine all of the forces of nature into one single theory. So, the questions we want to ask are what do these dimensions look like, and what do the resulting 4D theories look like after dimensional reduction? To answer the first question, we must first introduce Calabi-Yau manifolds to consider an extra 6 dimensional space, and if we want to consider a 7-dimensional space then we must study G_2 manifolds, which we introduce in Section 4. For the second question, we approach the answer in Sections 3 and 4 for Calabi-Yau and G_2 manifolds respectively. The hope is that a reduction of these theories will give us something that looks like a unified theory of physics in 4D, with the ultimate goal being a unification of gravity with the standard model. These questions are the driving motivation for this thesis.

Much of the introductory content for this section shall be taken from [5] and [14], as well as drawing from [4]. I shall do my best to highlight important aspects and give a more verbose explanation of the more important concepts required for this thesis.

2.1 Complex, Hermitian and Kähler Manifolds

Before defining a Calabi-Yau manifold, we must first introduce various definitions and notions that we have not yet covered. A Calabi-Yau is a complex manifold with certain properties, and so firstly we should define precisely what we mean by a complex manifold. We will simply copy the definition from [4], as it is very similar to the definition of a real manifold.

Definition 2.1 (Complex Manifold)**.** *Let M be a 2n-dimensional real manifold with open covering* $\{U_i\}$*. Similarly to a real manifold, we define a coordinate chart* (U_i, ψ_i) *where* $\psi_i : U_i \to \mathbb{C}^n$ *is an homeomorphism from from* U_i *to an open subset of* $\mathbb{C}^n$*. We say that* $(M, \{U_i, \psi_i\})$ *is a complex manifold of complex dimension n if the transition functions* $\psi_{ij} = \psi_j \circ \psi_i^{-1} : \psi_i(U_i \bigcap U_j) \to \psi_j(U_i \bigcap U_j)$ *are all holomorphic functions. What this means is that M is a complex manifold if locally it looks like* $\mathbb{C}^n$ *[4].*

Now that we've introduced a complex manifold in this way, we notice that this is an incredibly similar definition to how a real manifold is defined, just with complex spaces and holomorphic functions instead of real spaces and differentiable functions. We shall usually just assume that a manifold is complex throughout the thesis (where appropriate) as opposed to showing it directly as we wish to avoid proofs that require in-depth real or complex analysis in this thesis, as they can become too advanced for what we wish to cover here.

Given local coordinates $z^\mu, z^{\bar{\mu}}$ on a complex manifold M corresponding to holomorphic and anti-holomorphic coordinates, where $z^{\bar{\mu}} = \bar{z}^\mu$, we can split the basis of tangent vectors into $\{\frac{\partial}{\partial z^\mu}\}, \{\frac{\partial}{\partial z^{\bar{\mu}}}\}$ corresponding to the tangent spaces T_pM^+ and T_pM^- respectfully [14]. These combine to give $T_pM = T_pM^+ \otimes T_pM^-$. We have the same for cotangent vectors and the cotangent spaces, with two bases $\{dz^\mu\}, \{dz^{\bar{\mu}}\}$, such that T_p^*M is spanned by $\{dz^\mu, dz^{\bar{\mu}}\}$. The basis of the cotangent space is what we will use to construct complex differential forms, which we will come to soon.

We continue by introducing the almost complex structure of a (real or complex) manifold, which shall contribute extensively, yet subtly, to this thesis. Essentially, it is just a tensor in local coordinates that squares to the minus Identity matrix.

Definition 2.2 (Almost complex structure (ACS))**.** *We can define the almost complex structure on a 2n-dimensional manifold M as follows: [5]*

$$J_m^n = idz^\mu \otimes \frac{\partial}{\partial z^\mu} - idz^{\bar{\mu}} \otimes \frac{\partial}{\partial z^{\bar{\mu}}} \tag{76}$$

where we have that m,n can be 'holomorphic' or 'anti-holomorphic' indices, i.e., μ *or* $\bar{\mu}$ *respectfully. For example [5]*

$$J_\mu{}^\nu = i\delta_\mu{}^\nu,\ J_{\bar{\mu}}{}^{\bar{\nu}} = -i\delta_{\bar{\mu}}{}^{\bar{\nu}},\ J_\mu{}^{\bar{\nu}} = 0 \tag{77}$$

To see why we get these, we consider how the different choices of m,n would arise in Equation (76). We can then write J in matrix form as

$$J = \begin{pmatrix} i\mathbb{I}_n & 0 \\ 0 & -i\mathbb{I}_n \end{pmatrix} \tag{78}$$

where $\mathbb{I}_n$ is the n dimensional identity matrix. Thus, squaring J we get [14]

$$J^2 = -\mathbb{I}_{2n} \tag{79}$$

or in index notation [5]

$$J_k{}^i J_i{}^m = -\delta_k{}^m \tag{80}$$

Note that every 2n-dimensional manifold admits an almost complex structure locally, but only on complex manifolds is it defined globally [14].

Lemma 2.1 (ACS is real)**.** *While J is written in terms of complex coordinates and contains imaginary numbers, it actually turns out to be real [5]*

$$\bar{J} = -idz^{\bar{\mu}} \otimes \frac{\partial}{\partial z^{\bar{\mu}}} + idz^{\mu} \otimes \frac{\partial}{\partial z^{\mu}} = J \tag{81}$$

Definition 2.3 (Hermitian Manifold)**.** *If M is a complex manifold that admits a metric of the form*

$$ds^2 = g_{\mu\bar{\nu}} dz^{\mu} dz^{\bar{\nu}} \tag{82}$$

then we say that M is an Hermitian manifold. Note that a complex manifold always admits an Hermitian metric, so when we say that M is Hermitian, we actually mean that the pair (M,g) is Hermitian, where g is of the form given above [5]. Note that by this definition the metric for an Hermitian manifold has [5]

$$g_{\mu\nu} = g_{\bar{\mu}\bar{\nu}} = 0 \tag{83}$$

Lemma 2.2. *Let M be an Hermitian manifold. Then the metric satisfies [5]*

$$g_{ij} = J_i{}^m J_j{}^n g_{mn} \tag{84}$$

This is essentially just contraction of indices to write the metric in terms of the ACS.

Theorem 2.1 (ACS as a 2-form)**.** *Let M be an Hermitian manifold. Then the ACS is a 2-form: [5]*

$$J_{mn} = -J_{nm} \tag{85}$$

Proof: The proof of this is given in [5], but not in very much detail, so we shall be more explicit. Writing our metric in the form given in Lemma 2.2 and multiplying by $J_k{}^i$ we get

$$J_k{}^i g_{ij} = J_k{}^i J_i{}^m J_j{}^n g_{nm} \tag{86}$$

where we swapped the indices of the metric on the RHS. Now we can contract indices with the metric on both sides and using the identity that $J^2 = -I$ on the right hand side to get

$$J_{kj} = -\delta_k{}^m J_j{}^n g_{nm} \tag{87}$$

$$\Rightarrow J_{kj} = -J_{jk} \tag{88}$$

which, as J is a tensor, means that for an Hermitian manifold the ACS defines a 2-form [5].

Note that we have not yet distinguished how forms differ on complex manifolds to those on real manifolds. We can now split the forms up into holomorphic and anti-holomorphic components but as these components form a basis, complex forms behave in a similar way to real forms. Complex forms are explained in [14] better than [5] as it goes into more depth and states things more explicitly, so we'll follow their argument for this discussion.

Definition 2.4 (Complex Forms)**.** *Given a r-form F on a complex manifold that consists of p holomorphic cotangent vectors and q anti-holomorphic cotangent vectors such that $p+q=r$, we can write F as [14]*

$$F = F_{\mu_1..\mu_p\bar{\nu}_1..\bar{\nu}_q} dz^{\mu_1} \wedge .. \wedge dz^{\mu_p} \wedge dz^{\bar{\nu}_1} \wedge .. \wedge dz^{\bar{\nu}_q} \tag{89}$$

with $F \in \Omega^{(p,q)}(M)$, and will refer to F as a (p,q)-form.

We can write the ACS in a way that will prove incredibly useful for us later on. This form is used in [5] without being explicitly stated or proved, so we will prove it ourselves.

Lemma 2.3 (Rewriting J as a form)**.** *Now that we've seen that J is a 2-form, we shall write it as a (1,1)-form: [14]*

$$J = J_{\mu\bar{\nu}} dz^{\mu} \wedge dz^{\bar{\nu}} \tag{90}$$

where

$$J_{\mu\bar{\nu}} = i g_{\mu\bar{\nu}} \tag{91}$$

Proof: *We can use the identity*

$$J_{\mu}^{\ \lambda} = i\delta_{\mu}^{\ \lambda} \tag{92}$$

from Equation (77) and by right-multiplication of $g_{\lambda\bar{\nu}}$ we get

$$J_{\mu}^{\ \lambda} g_{\lambda\bar{\nu}} = i\delta_{\mu}^{\ \lambda} g_{\lambda\bar{\nu}} \tag{93}$$

which by contraction of indices gives us

$$J_{\mu\bar{\nu}} = i g_{\mu\bar{\nu}} \quad \square \tag{94}$$

This Lemma essentially allows us to consider the metric as a form - by multiplying an Hermitian metric by i and then taking the complex conjugate we have essentially asymmetrized the metric, and thus get a differential form.

Before introducing the incredibly important definition of Kähler manifolds, we should introduce something called the Dolbeault operators which act on complex forms. They are a way of writing the exterior derivative of a complex form in terms of holomorphic and anti-holomorphic coordinates:

Definition 2.5 (Dolbeault operators)**.** *We define the Dolbeault operators and their action on a complex form F as [14]*

$$d = \partial + \bar{\partial} \tag{95}$$

such that

$$\partial F = \frac{\partial F_{\mu_1..\mu_p\bar{\nu}_1..\bar{\nu}_q}}{\partial z^\lambda} dz^\lambda \wedge dz^{\mu_1} \wedge .. \wedge dz^{\mu_p} \wedge dz^{\bar{\nu}_1} \wedge .. \wedge dz^{\bar{\nu}_q} \tag{96}$$

and

$$\bar{\partial} F = (-1)^p \frac{\partial F_{\mu_1..\mu_p\bar{\nu}_1..\bar{\nu}_q}}{\partial z^{\bar{\lambda}}} dz^{\mu_1} \wedge .. \wedge dz^{\mu_p} \wedge dz^{\bar{\lambda}} \wedge dz^{\bar{\nu}_1} \wedge .. \wedge dz^{\bar{\nu}_q} \tag{97}$$

We then have that $\partial F \in \Omega^{(p+1,q)}(M)$ and $\bar{\partial} F \in \Omega^{(p,q+1)}$. These Dolbeault operators individually act like the exterior derivative does with regard to any aforementioned properties.

Thinking back to de Rham cohomology from Subsection 1.2, how should we think about the cohomology of a complex form with respect to these new operators? Well, we can introduce a new type of cohomology called Dolbeault cohomology:

Definition 2.6 (Dolbeault Cohomology)**.** *We define the ∂- and $\bar{\partial}$-cohomology groups similarly to the de Rham cohomology groups: [4]*

$$\mathbb{H}_\partial^{(p,q)} = \mathbb{C}_\partial^{(p,q)} / \mathbb{B}_\partial^{(p,q)} \tag{98}$$

and similarly for the $\bar{\partial}$-cohomology group, where

$$\mathbb{C}_\partial = \{F \in \Omega^{(p,q)}(M) | \partial F = 0\} \tag{99}$$

$$\mathbb{B}_\partial = \{F \in \Omega^{(p,q)}(M) | F = \partial A\} \tag{100}$$

with similar definitions for $\bar{\partial}$.

Theorem 2.2 (Dolbeault Cohomology $\cong$ de Rham Cohomology)**.** *We have that [5]*

$$\partial\partial^\dagger + \partial^\dagger\partial = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial} = \frac{1}{2}(dd^\dagger + d^\dagger d) \tag{101}$$

so therefore Dolbeault Cohomology and de Rham Cohomology are equivalent. This is because if a form in de Rham cohomology vanishes under the Laplacian, then it will for both the ∂ Laplacian and the $\bar{\partial}$ Laplacian.

Definition 2.7 (Hodge Diamond)**.** *The Hodge numbers of a complex manifold are defined in a very similar way to the Betti numbers were in Definition 1.18. The (p,q)-th Hodge number is given by: [4]*

$$b_{pq} = dim \mathbb{H}_{\bar{\partial}}^{(p,q)} \tag{102}$$

$$= dim \mathbb{H}_{\partial}^{(p,q)} \tag{103}$$

They are usually arranged into what we call a Hodge Diamond; for the 3-complex-dimensional case we get [4]

$$\begin{pmatrix} & & & b_{00} & & & \\ & & b_{10} & & b_{01} & & \\ & b_{20} & & b_{11} & & b_{02} & \\ b_{30} & & b_{21} & & b_{12} & & b_{03} \\ & b_{31} & & b_{22} & & b_{13} & \\ & & b_{32} & & b_{23} & & \\ & & & b_{33} & & & \end{pmatrix} \tag{104}$$

Lemma 2.4 (Betti Numbers of Complex Manifold)**.** *Given the Hodge numbers b_{pq} of a complex manifold M, we can obtain it's Betti numbers in a simple way: [14]*

$$b_k(M) = \sum_{k=p+q} b_{pq} \tag{105}$$

That is, the k-th Betti number is just the sum of the k-th row of the Hodge diamond. This makes sense, as the Hodge numbers were obtained by finding harmonic forms with respect to the Dolbeault operators, which were just a way of decomposing the exterior derivative.

With these new tools, we can define a Kähler manifold, which takes us one step closer to defining a Calabi-Yau manifold. As we will see later, Calabi-Yau manifolds are just Ricci-flat Kähler manifolds, so studying Kähler geometry is of crucial importance to us. Our definition of Kähler manifolds comes directly from [5].

Definition 2.8 (Kähler Manifold)**.** *Let M be an Hermitian manifold with almost complex structure J. We call M a Kähler manifold if*

$$dJ = 0 \tag{106}$$

where we now refer to J as the Kähler form [5]. Using Dolbeault operators, we require [5]

$$\begin{aligned} dJ &= (\partial + \bar{\partial})J && (107)\\ &= i\frac{\partial g_{\mu\bar{\nu}}}{\partial z^{\lambda}} dz^{\lambda} \wedge dz^{\mu} \wedge dz^{\bar{\nu}} - i\frac{\partial g_{\mu\bar{\nu}}}{\partial z^{\bar{\lambda}}} dz^{\mu} \wedge dz^{\bar{\lambda}} \wedge dz^{\bar{\nu}} = 0 && (108) \end{aligned}$$

which means that

$$\frac{\partial g_{\mu\bar{\nu}}}{\partial z^{\lambda}} = \frac{\partial g_{\mu\bar{\nu}}}{\partial z^{\bar{\lambda}}} = 0 \tag{109}$$

vanish separately, as $\partial J, \bar{\partial} J$ exist in separate spaces: the space of (2,1)-forms and (1,2)-forms, respectively.

Lemma 2.5 (Kähler form is not exact)**.** *Assume that M is a compact Kähler manifold without boundary. We have that the volume form is proportional to $J \wedge ... \wedge J$ [5], so if we assume that J is exact, i.e., $J = dA$, then we have [5]*

$$\begin{aligned} Vol(M) &= \int J \wedge ... \wedge J && (110)\\ &= \int_M dA \wedge ... \wedge J && (111)\\ &= \int_M d(A \wedge J \wedge ... \wedge J) - A \wedge d(J \wedge ... \wedge J) && (112)\\ &= \int_{\partial M} A \wedge J \wedge ... \wedge J = 0 && (113) \end{aligned}$$

where we have used Lemma 1.7 and the fact that J is closed in Equation (112), and Theorem 1.4 in Equation (113). However, obviously M cannot have null volume, therefore J is not exact. If M did have a boundary, then this integral would not necessarily vanish, so therefore $b_{10} = b_{01} = 0$ for Kähler manifolds (see Appendix A for an explanation of this).

Theorem 2.3 (J is Harmonic)**.** *We defined harmonic forms to be those that are both closed and co-closed. It can be shown that the Kähler form J is not just closed, but also co-closed [5], and from Lemma 2.5 we have shown that it is not exact. Therefore J is a harmonic form. Therefore* $b_{11} \geq 1$ *for a Kähler manifold.*

We would like to introduce the Ricci form, a form associated to the Riemann Tensor of the manifold. This is not a form that we actually pay much attention to, as the manifolds that we are considering are usually Ricci-flat. Thus we introduce the form in hopes of showing that it vanishes (or is exact) in most cases.

Definition 2.9 (Ricci Form)**.** *Let (M,g) be a Kähler manifold with Ricci Tensor* R_{mn} *corresponding to the metric g. Then we define the Ricci Form as [5]*

$$R(g) = iR_{\mu\bar{\nu}} dz^{\mu} \wedge dz^{\bar{\nu}} \tag{114}$$

which is a (1,1)-form. For Kähler manifolds we obtain the Ricci Tensor in the same way as would for a manifold with Levi-Civita connection [5], but for general Hermitian manifolds we would need to introduce the more general Hermitian connection and be more careful with how we compute the connection coefficients [14].

We will not introduce the Hermitian connection here as we will only be considering examples of Kähler manifolds and thus will be assuming that our connection is Levi-Civita. Thankfully, the following theorem simplifies the calculation of the Ricci Form for Hermitian manifolds so that we would not have to worry about dealing with the Hermitian connection.

Theorem 2.4 (Ricci Form of an Hermitian Manifold)**.** *A convenient form of the Ricci form for a Hermitian manifold (M,g) in terms of the Dolbeault operators is [5]*

$$R(g) = i\partial\bar{\partial} log(g^{\frac{1}{2}}) \tag{115}$$

where g is the determinant of the Hermitian metric. Naturally, we can also write the Ricci Form of a Kähler manifold in this way.

We introduce a cohomology class known as the First Chern Class as it will be important in discussing Ricci-Flat Kähler manifolds (Calabi-Yau manifolds). Our definition combines information from both [5] and [14].

Definition 2.10 (First Chern Class)**.** *Let M be a Kähler manifold that admits a metric g with corresponding Ricci form R(g). Then we define a cohomology class of M, known as the First Chern Class* $c_1(M)$*, as*

$$c_1 = [R(g)] \tag{116}$$

such that for another Ricci form $R(h) \in c_1(M)$ *we have*

$$R(g) \sim R(h) + dA \tag{117}$$

where dA is an exact form [5]. That is, the first Chern class of a Kähler manifold (M,g) is the cohomology group of Ricci forms on M that differ by an exact form [14].

We have really only scratched the surface of Chern classes here - further discussion can be found in [4]. However, when dealing with Calabi-Yau manifolds we will only really be interested in the First Chern Class.

Definition 2.11 (Ricci-Flat Metric)**.** *Let (M,g) be a Kähler manifold with Ricci Form R(g). If*

$$R(g) = 0 \tag{118}$$

i.e. the form vanishes, then we say that g is a Ricci-flat metric [14].

2.2 Defining a Calabi-Yau

So far we have studied complex geometry up to the point of considering Kähler manifolds, i.e. a complex manifold with a closed ACS that admits an Hermitian metric. The purpose for introducing the Kähler manifold is because Calabi-Yau's are a special case of these, satisfying many different but equivalent definitions.

Our reason for studying Calabi-Yau manifolds is that they are the suitable candidate for the extra dimensions of a ten-dimensional string theory, and so to extend our original Kaluza-Klein theory to compactifications of string theories it is necessary to become very familiar with Calabi-Yau's.

Definition 2.12 (Calabi-Yau Manifold)**.** *Let (M,g) be a compact Kähler manifold. If g is a Ricci-flat metric, then M is a Calabi-Yau manifold [4].*

The question 'does a Kähler manifold with vanishing first Chern class admit a Ricci-flat metric?' is a very difficult problem to prove. The affirmative was conjectured by Calabi, known as the Calabi Conjecture, and was proved by Yau's Theorem [4]. We won't cover this theorem's proof, important as it is, but we can show the converse; This proof is shown in [5] but it is an interesting proof so we shall include it here.

Theorem 2.5. *Let M be a Kähler manifold admitting a Ricci-flat metric g: $R(g) = 0$. Then the manifold M has vanishing first Chern class:*

$$c_1 = 0 \tag{119}$$

Proof: *Recalling Definition 2.10, we can define $c_1(M) = [R(g)]$. As this is a cohomology class, two elements $R(g), R(h) \in c_1$ are related by*

$$R(h) \sim R(g) + dA \tag{120}$$

but we have said that g has $R(g) = 0$, so we get that

$$R(h) = dA \tag{121}$$

and as this form is then exact, we get that c_1 is a trivial class, i.e. $c_1 = 0$ [5]. What we mean by this is that every Ricci form in $c_1(M)$ is exact, and so the cohomology class is empty by recalling Definition 1.17.

Theorem 2.6. *An n-complex-dimensional Kähler M is a Calabi-Yau iff it admits a unique holomorphic (n,0)-form that is globally defined and does not vanish anywhere [4]. We can write this form as [5]*

$$\Omega = \Omega_{\mu_1 \dots \mu_n} dz^{\mu_1} \wedge \dots \wedge dz^{\mu_n} \tag{122}$$

Proof: A sketch of this proof is given in [5], which we shall include here as it allows us to see why Theorem 2.4 is so powerful. If we find the norm of this form:

$$|\Omega|^2 = \Omega_{\mu_1 .. \mu_n} \bar{\Omega}^{\mu_1 .. \mu_n} \tag{123}$$

where we set

$$\Omega_{\mu_1 .. \mu_n} = f(z) \varepsilon_{\mu_1 .. \mu_n} \tag{124}$$

$$\bar{\Omega}^{\mu_1 .. \mu_n} = \bar{f}(z) \varepsilon_{\bar{\nu}_1 .. \bar{\nu}_n} g^{\mu_1 \bar{\nu}_1} .. g^{\mu_n \bar{\nu}_n} \tag{125}$$

$$= \bar{f}(z) g^{-\frac{1}{2}} \varepsilon^{\mu_1 .. \mu_n} \tag{126}$$

which then gives us

$$|\Omega|^2 = g^{-\frac{1}{2}} |f|^2 \tag{127}$$

$$\Rightarrow g^{\frac{1}{2}} = \frac{|f|^2}{|\Omega|^2}(z) \tag{128}$$

where $g^{\frac{1}{2}}$ is now a coordinate scalar. Then, using Theorem 2.4, we get

$$R = i\partial\bar{\partial} \log \frac{|f|^2}{|\Omega|^2}(z) \tag{129}$$

where we've thus shown that R is exact and thus $c_1 = 0$ from Theorem 2.5. This is because if there exists $R = dA \in c_1(M)$ then there must also exist a Ricci flat R(g). Therefore the existence of the (n,0)-form defined above gives us an equivalent definition of a Calabi-Yau manifold.

Next we can discuss the cohomology of Calabi-Yau manifolds, as this is quite revealing about the nature of individual examples. For this discussion we'll only consider Calabi-Yau manifolds of complex dimension 3 (real dimension 6), as these are the ones of interest to us later in the string theory reductions that we will consider.

Theorem 2.7 (Hodge Diamond of a Calabi-Yau)**.** *Let M be a Calabi-Yau manifold of complex dimension 3. Then we know from Theorem 2.6 that it admits a holomorphic (3,0)-form, as well as a Kähler (1,1)-form. The Hodge numbers have the Hodge duality [4]*

$$b_{pq} = b_{(n-p),(n-q)} \tag{130}$$

which is due to the unique mapping of the Hodge star. We can also complex conjugate a complex form uniquely such that

$$b_{pq} = b_{qp} \tag{131}$$

There's also the holomorphic duality: [5]

$$b_{0q} = b_{0,(n-q)} \tag{132}$$

The consequences of this for the Hodge diamond is that we get a vertical and horizontal symmetry, i.e.

$$\begin{pmatrix} & & & b_{00} & & & \\ & & b_{10} & & b_{01} & & \\ & b_{20} & & b_{11} & & b_{02} & \\ b_{30} & & b_{21} & & b_{12} & & b_{03} \\ & b_{31} & & b_{22} & & b_{31} & \\ & & b_{32} & & b_{23} & & \\ & & & b_{33} & & & \end{pmatrix} \tag{133}$$

becomes

$$\begin{pmatrix} & & & b_{00} & & & \\ & & b_{10} & & b_{10} & & \\ & b_{10} & & b_{11} & & b_{10} & \\ b_{30} & & b_{21} & & b_{21} & & b_{30} \\ & b_{10} & & b_{11} & & b_{10} & \\ & & b_{10} & & b_{10} & & \\ & & & b_{00} & & & \end{pmatrix} \tag{134}$$

As b_{00} just counts the number of connected components of the manifold due to counting the number of independent scalars on the manifold, we have that b_{00} = 1. From Lemma 2.5 we have also that $b_{10} = 0$, as well as $b_{30} = 1$ due to the holomorphic (3,0)-form being unique. So finally we have the diamond

$$\begin{pmatrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & b_{11} & & 0 & \\ 1 & & b_{21} & & b_{21} & & 1 \\ & 0 & & b_{11} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{pmatrix} \tag{135}$$

which means that for a Calabi-Yau of complex dimension 3 we have it's topology being determined by just two numbers, b_{11} and b_{21}. Bear in mind that we have already showed that every Calabi-Yau has $b_{11} \geq 1$ due to the existence of at least one Kähler form that corresponds to a Ricci-flat metric. Thus, b_{11} counts the number of Ricci-flat metrics on the Calabi-Yau. Note that this form of the Hodge Diamond is only valid for Calabi-Yau manifolds that are simply connected [2]. We will see an example of a non-simply connected Calabi-Yau and consider it's Hodge numbers in Example 2.1.

Corollary 2.1 (Euler Characteristic of Calabi-Yau Manifolds)**.** *The Euler Characteristic of simply-connected Calabi-Yau M, $\chi(M)$, takes the value [4]*

$$\chi(M) = 2(b_{11} - b_{21}) \tag{136}$$

Proof: *Recall Theorem 1.5, and Lemma 2.4 such that we get*

$$\chi(M) = 2b_0 - 2b_1 + 2b_2 - b_3 \tag{137}$$
$$= 2(1) - 2(0+0) + 2(0+b_{11}+0) - (1+b_{21}+b_{21}+1) \tag{138}$$
$$= 2(b_{11} - b_{21}) \quad \square \tag{139}$$

where in Equation (138) we used (135).

2.3 Examples of Calabi-Yau manifolds

When trying to construct examples of Calabi-Yau manifolds, there is an issue - trying to write the metrics explicitly is not possible in many cases. Even Yau's theorem, which proved the Calabi-Yau conjecture, is just an existence theorem as opposed to providing the actual metric. The examples which we'll consider in this subsection are chosen so that they're simple enough to be able to consider their metrics explicitly.

A very useful example of Calabi-Yau manifolds which are often used in string theory are constructed via orbifolds. We can define a complex orbifold in a very simple way, and then we can give some examples and show whether or not they are Calabi-Yau manifolds. [14] gives the most simple definition, but [10] and [2] give more comprehensive details which we will use a lot for our discussion. Firstly, we will consider a non-orbifold example that will be useful when we do construct orbifolds:

Example 2.1 (6-Torus)**.** *The 6-Torus is a 3-complex-dimensional (and thus 6-real-dimensional) manifold, defined by the identifications*

$$z^\mu \sim z^\mu + 1 \sim z^\mu + i \tag{140}$$

where $z^\mu \in \mathbb{C}$ such that $(z^\mu) \in \mathbb{C}^3$. This definition of a torus should be familiar to readers. Naturally, a torus is compact, so all we need to show to prove T^6 is a Calabi-Yau is that it is Kähler and that it is Ricci-flat. Note that the metric for $\mathbb{C}^3 \cong \mathbb{R}^6$ is just

$$ds^2 = (dx^\mu)^2 + (dy^\mu)^2 \tag{141}$$

so if we define $z^\mu = x^\mu + iy^\mu$, such that $dz^\mu = dx^\mu + idy^\mu, dz^{\bar{\mu}} = dx^{\bar{\mu}} - idy^{\bar{\mu}}$, then we can write Equation (141) as

$$ds^2 = dz^\mu dz^{\bar{\mu}} \tag{142}$$
$$= (dx^\mu + idy^\mu)(dx^{\bar{\mu}} - idy^{\bar{\mu}}) \tag{143}$$
$$= (dx^\mu)^2 + (dy^\mu)^2 \tag{144}$$

which matches the metric for $\mathbb{C}^3$. By making the identifications in Equation (140) to define T^6, we do not affect the metric and so T^6 can inherit this induced metric from $\mathbb{C}^3$. Therefore, considering Definition 2.3, we have shown that T^6 with the metric above is Hermitian, such that

$$ds^2 = g_{\mu\bar{\nu}} dz^\mu dz^{\bar{\nu}} = \delta_{\mu\bar{\nu}} dz^\mu dz^{\bar{\nu}} \tag{145}$$

To extend this to being Kähler, we need to define the ACS (or Kähler form) and show it's closed. We've seen in Lemma 2.3 that we can define the Kähler form on an Hermitian manifold by

$$J = J_{\mu\bar{\nu}} dz^\mu \wedge dz^{\bar{\nu}} \tag{146}$$

$$= i g_{\mu\bar{\nu}} dz^\mu \wedge dz^{\bar{\nu}} \tag{147}$$

$$= i \delta_{\mu\bar{\nu}} dz^\mu \wedge dz^{\bar{\nu}} \tag{148}$$

which we can then take the exterior derivative of

$$dJ = (\partial + \bar{\partial})J = i(\partial + \bar{\partial})\delta \tag{149}$$

$$= i\frac{\partial \delta_{\mu\bar{\nu}}}{\partial z^\lambda} dz^\lambda \wedge dz^\mu \wedge dz^{\bar{\nu}} + i\frac{\partial \delta_{\mu\bar{\nu}}}{\partial z^{\bar{\lambda}}} dz^{\bar{\lambda}} \wedge dz^\mu \wedge dz^{\bar{\nu}} \tag{150}$$

$$= 0 \tag{151}$$

as clearly δ vanishes under a derivative. Therefore we have shown that T^6 is a Kähler manifold. All we need to do to prove it is a Calabi-Yau is to show it's Ricci-flat, which in this case is very simple. Recall Theorem 2.4, and recognise that the determinant of our metric is $det\delta_{\mu\bar{\nu}} = 1$, then

$$R = i\partial\bar{\partial} log 1 = 0 \tag{152}$$

Thus T^6 is a Ricci-flat Kähler manifold, and therefore a Calabi-Yau manifold of complex dimension 3.

Now that we've shown that T^6 is Calabi-Yau, we would like to consider it's Hodge numbers. Note that in Theorem 2.7 we said that for non-simply connected Calabi-Yau manifolds we do not have the usual Hodge diamond as in Theorem 2.7, so we'll have to see what the Hodge numbers are for a Calabi-Yau in this case. We know from Corollary 2.3 that $b_{11} \geq 1$, so all we need to do is determine if there are any more harmonic (1,1)-forms for T^6. Consider the basis of harmonic (1,1)-forms:

$$\{\omega_{\mu\bar{\nu}} = dz^\mu \wedge dz^{\bar{\nu}} | \triangle\omega = 0\} \tag{153}$$

We could pick $\binom{3}{1}$ indices for μ and the same for $\bar{\nu}$. Therefore we have

$$b_{11} = \binom{3}{1} * \binom{3}{1} = 9 \tag{154}$$

We can obtain $b_{21} = b_{12}$ by following the same logic as b_{11}: we have basis of harmonic (1,2)-forms

$$\{\omega_{\mu\bar{\nu}\bar{\lambda}} = dz^\mu \wedge dz^{\bar{\nu}} \wedge dz^{\bar{\lambda}} | \triangle\omega = 0\} \tag{155}$$

Then we have again $\binom{3}{1}$ choices of indices for μ and $\binom{3}{2}$ indices for $\bar{\nu}, \bar{\lambda}$. Therefore

$$b^{12} = \binom{3}{1} * \binom{3}{2} = 9 \tag{156}$$

For a simply-connected Calabi-Yau we would then be able to write the Hodge diamond as

$$\begin{pmatrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 9 & & 0 & \\ 1 & & 9 & & 9 & & 1 \\ & 0 & & 9 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{pmatrix} \tag{157}$$

but as the Torus is not simply-connected, we have to work around this. It is worth noting that some authors require simply-connectedness in the definition of Calabi-Yau manifolds. Using the same logic as the calculation of b_{11}, b_{12}, we can thus say that

$$b_{pq} = \binom{3}{p} * \binom{3}{q} \tag{158}$$

such that the Betti numbers, from Lemma 2.4, end up being

$$b_k = \binom{6}{k} \tag{159}$$

Therefore the Hodge diamond of the 6-Torus looks like two Pascal triangles connected together. We can write the (3-0)-form of T^6 as

$$\Omega = \varepsilon_{\mu\nu\rho} dz^{\mu} \wedge dz^{\nu} \wedge dz^{\rho} \tag{160}$$

which is clearly harmonic, and provides an alternative proof that T^6 is a Calabi-Yau.

Now that we've seen the example of a non-simply connected Calabi-Yau, it would be good to find an example that has the Hodge diamond as given in Theorem 2.7. To do this, we can use our T^6 example and create an orbifold.

Definition 2.13 (Complex Orbifold)**.** *Let M be a complex manifold and G be a discrete group, then*

$$\Gamma = M/G \tag{161}$$

is a complex orbifold [14].

Theorem 2.8 (Orbifold Singularities)**.** *Let $\Gamma = M/G$ be a complex orbifold. If nontrivial group elements of G leave points in M invariant, then these points are orbifold singularities [2]. That is, for each $z \in M$, if we define the stabilizer subgroup to be*

$$Stab(z) = \{g \in G | g \cdot z = z\} \tag{162}$$

such that if $Stab(z) \neq \{1\}$ then z is a singular point of the orbifold [10]. Thus, clearly, if a form in a cohomology group of M is not invariant under G then it is not in the cohomology group of the orbifold.

Example 2.2 (One-dimensional Orbifold)**.** *We can begin by looking at one-dimensional examples of complex orbifolds. A natural choice of 1-D complex manifold M is $\mathbb{C}$, the complex plane. We can then pick G to be $\mathbb{Z}_n$, i.e.*

$$\mathbb{Z}_n = \{e^{\frac{2ki\pi}{n}} | k \in \{1,..,n\}\} \tag{163}$$

which is just the finite set of rotations in the plane by an angle $\frac{2\pi}{n}$. By taking this quotient we are saying that for $z \in \mathbb{C}$ we get

$$z \sim e^{\frac{2i\pi}{n}} z \tag{164}$$

For the case of $\mathbb{Z}_2$ this would mean $z \sim -z$, and thus we end up with a cone [2]. We can calculate the singularities of this by saying

$$e^{\frac{2i\pi}{n}} z = z \tag{165}$$

which only has solution z=0. Therefore for all orbifolds $\mathbb{C}/\mathbb{Z}_n$, we get a singularity at the origin. The orbifold is technically no longer a manifold as it is no longer smooth. Therefore this cannot be a Calabi-Yau, as well as the fact that it is non-compact. With this being said, we can smooth out singularities of orbifolds using Eguchi-Hanson spaces to recover manifolds, which we'll consider in Subsection 2.4.

Example 2.3 (6-Torus Z_3 Orbifold)**.** *As mentioned previously, we would like to see an example of a simply-connected Calabi-Yau, so that we get the Hodge diamond as in Theorem 2.7. To do this, we can take Example 2.1 and use this as our manifold M in Definition 2.13.*

If we take our quotient group to be $G = \mathbb{Z}_3$ then we get an additional identification of

$$z^\mu \sim e^{\frac{2i\pi}{3}} z^\mu \tag{166}$$

from which we can now use Theorem 2.8 to calculate the singularities of the orbifold.

By looking at the singularities from the solution to Exercise 9.2 from [2], we can consider the case of T^6/Z_3 instead of T^4/Z_3. That is, let our singularities take the form

$$z^\mu = \frac{c}{\sqrt{3}} e^{\frac{\pi i}{6}} = c(\frac{1}{2} + \frac{i}{2\sqrt{3}}) \tag{167}$$

for $c \in \{0,1,2\}$, then we get singularities [5]. c=0 is just the singularity at $z^\mu = 0$, like in the 1-complex-dimensional case in Example 2.2. Note that [2], [10], [14], and [5] all follow this example, but none of them explicitly show why these are singularities. We shall do this now, by *letting z^μ in Equation (166) transform under $g = e^{\frac{2\pi i}{3}}$:*

$$e^{\frac{2\pi i}{3}} z^\mu = \frac{c}{\sqrt{3}} e^{\frac{2\pi i}{3}} e^{\frac{\pi i}{6}} \tag{168}$$

$$= \frac{c}{\sqrt{3}} e^{\frac{5\pi i}{6}} \tag{169}$$

$$= \frac{c}{\sqrt{3}}(-\frac{\sqrt{3}}{2} + \frac{i}{2}) \tag{170}$$

$$= c(-\frac{1}{2} + \frac{i}{2\sqrt{3}}) \tag{171}$$

$$= c(-\frac{1}{2} + \frac{i}{2\sqrt{3}}) + 1 + ... + 1 \tag{172}$$

$$= c(\frac{1}{2} + \frac{i}{2\sqrt{3}}) \tag{173}$$

$$= z^\mu \tag{174}$$

where in Equation (172) we used the fact that for T^6 we have the identification $z^\mu \sim z^\mu + 1$ enough times to get a +c factor to arrive at the conclusion that for this choice of z^μ we get that $gz^\mu = z^\mu$, and thus by Theorem 2.8 they are singularities. As each $z^\mu \in \mathbb{C}$ for $(z^\mu) \in \mathbb{C}^3$ can take one of these 3 values of c, and we have 3 coordinates to choose from, we get $3^3 = 27$ singularities [5].

To show that the Hodge numbers of this orbifold match that of Theorem 2.7, we can consider whether various forms are invariant under the action of $\mathbb{Z}_3$. For example:

$$dz^\mu \wedge dz^{\bar{\nu}} \to e^{\frac{2i\pi}{3}} dz^\mu \wedge e^{\frac{-2i\pi}{3}} dz^{\bar{\nu}} = dz^\mu \wedge dz^{\bar{\nu}} \tag{175}$$

so these are all preserved, leaving b_{11} unchanged. So, what we require generally is that a (p,q)-form has that:

$$\frac{2i\pi(p-q)}{3} = 2ni\pi,\ n \in \mathbb{Z} \tag{176}$$

which means that the only non-vanishing Hodge numbers are:

$$b_{00} = b_{33} = 1,\ b_{11} = 9,\ b_{30} = b_{03} = 1 \tag{177}$$

so the Hodge diamond of the $T^6/\mathbb{Z}_3$ orbifold is:

$$\begin{pmatrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 9 & & 0 & \\ 1 & & 0 & & 0 & & 1 \\ & 0 & & 9 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{pmatrix} \tag{178}$$

which matches the form of Theorem 2.7. Bear in mind, like in Example 2.2, we have to smooth the singularities for this to be considered a manifold, which we will do in Subsection 2.4.

As mentioned in the previous examples, the singularities of an orbifold mean it is not technically a manifold and thus can't be a Calabi-Yau unless we repair these singularities with Eguchi-Hanson spaces, which would affect the Hodge numbers. [2] explains that orbifold singularities are actually not an issue in string theory, as the strings can still propagate consistently on orbifolds despite their singularities so long as sufficient considerations are made. We would like to consider these Eguchi-Hanson spaces regardless, and see how we might go about constructing Calabi-Yau manifolds by the smoothing of complex orbifolds.

In particular, for the string theory reductions that we'll be considering in Section 3 we assume that the Euler characteristic is non-zero so that we can assume that only b_{11}, b_{12} are unique to our extra-dimensional manifolds. This assumption massively simplifies the reductions that we do, and this is why we care about the smoothing of the $T^6/\mathbb{Z}_3$ orbifold, as just T^6 would have other non-zero Hodge numbers.

2.4 Eguchi-Hanson Geometry

The n-complex-dimensional Eguchi-Hanson space, EH_n, gives us a way of smoothing orbifold singularities on manifolds. It is a convenient space to use as the metric is given explicitly, which is not always possible for Calabi-Yau manifolds, as mentioned previously.

Definition 2.14 (Eguchi-Hanson Metric)**.** *Let $w^i, w^{\bar{j}} \in \mathbb{C}^n$, and denote $\sigma = w_i w^{\bar{i}}$ as the magnitude of w^i [5]. Let c be some positive constant, then we can define the Eguchi-Hanson metric in n complex-dimensions as* [4] *[17] [5]*

$$g_{i\bar{j}} = (1+\frac{c}{\sigma^n})^{\frac{1}{n}}\{\delta_{i\bar{j}} - \frac{c w_i w_{\bar{j}}}{\sigma(c+\sigma^n)}\} \tag{179}$$

This metric on $\mathbb{C}^n$ then defines EH_n. The singularity at $\sigma^n = 0$ is just a coordinate singularity [5]. We have used the notation of [5] to write this metric, but have given it in the form written in [17].

Lemma 2.6 (EH_n Asymptotically Euclidean)**.** *Consider Definition 2.14 and let $\sigma \to \infty$. We see then that*

$$g_{i\bar{j}} \to \delta_{i\bar{j}} \tag{180}$$

Thus, we call EH_n asymptotically Euclidean, or rather, it is asymptotically $\mathbb{R}^{2n}$ [5].

Lemma 2.7 (EH_n Asymptotically Locally Euclidean)**.** *We require that metric this metric is one-to-one, but we see that w_i and $e^{\frac{2i\pi}{n}} w_i$ correspond to the same value of the metric [17]. Therefore, we make the following identification:*

$$w_i \sim e^{\frac{2i\pi}{n}} w_i \tag{181}$$

Then, considering Lemma 2.6, we now say that EH_n is asymptotically locally Euclidean (ALE). Thus, we get that EH_n is now asymptotically $\mathbb{R}^{2n}/\mathbb{Z}_n$ [17].

[4] The Wikipedia page for Eguchi-Hanson spaces contained a typo in this metric, which was corrected by the author.

Lemma 2.8 (Cohomology of EH_3). *We shall now consider the example of EH_3. From [2], we know that EH_n is Ricci-flat and Kähler, but as it is non-compact [5] it is not a Calabi-Yau manifold, and thus will not have a globally-defined holomorphic (n,0)-form. As we are using the metric to define the space, we know that this means*

$$b_{11} = 1 \tag{182}$$

from the corresponding Kähler form, and due to the $\mathbb{Z}_3$ identification made in Lemma 2.7 we know that the only remaining Hodge numbers will be the usual trivial Hodge numbers, i.e. the Hodge diamond is given by

$$\begin{pmatrix} 1 \\ 0\ \ 0 \\ 0\ \ 1\ \ 0 \\ 0\ \ 0\ \ 0\ \ 0 \\ 0\ \ 1\ \ 0 \\ 0\ \ 0 \\ 1 \end{pmatrix} \tag{183}$$

Proof: *We have already proved that the Hodge diamond will be in the form above for all Hodge numbers other than b_{11}. We wish to show that our Kähler form exists by using the metric given in Definition 2.14. Note that [5], [17], [2] do not show this, so we shall do it ourselves.*

Using Lemma 2.3, we write

$$J = J_{i\bar{j}}\, dw^i \wedge dw^{\bar{j}} = ig_{i\bar{j}}\, dw^i \wedge dw^{\bar{j}} \tag{184}$$

and from this we need to show that J is closed. Note that to use this Lemma we need $g_{i\bar{j}}$ to be Hermitian, so it's important to clarify this. We can write our metric in the line-segment form given in Definition 2.2:

$$ds^2 = (1+\frac{c}{\sigma^3})^{\frac{1}{3}}\{\delta_{i\bar{j}} - \frac{cw_i w_{\bar{j}}}{\sigma(c+\sigma^3)}\}dw^i dw^{\bar{j}} \tag{185}$$

so we thus see our metric is Hermitian. This is why we picked this form of the metric, similar to [17], as opposed to the ones given in [2] and [5]. Let us simplify the metric by defining

$$A(\sigma) = 1 + c\sigma^{-3} \tag{186}$$

such that

$$g_{i\bar{j}} = A^{\frac{1}{3}}\delta_{i\bar{j}} - c\sigma^{-4}A^{-\frac{2}{3}}w_i w_{\bar{j}} \tag{187}$$

Then we have

$$J = i[A^{\frac{1}{3}}\delta_{i\bar{j}} - c\sigma^{-4}A^{-\frac{2}{3}}w_i w_{\bar{j}}]dw^i \wedge dw^{\bar{j}} \tag{188}$$

To show that this form is closed, we should note some useful formulae:

$$\partial_k \sigma = \partial_k(w_i w^{\bar{i}}) \tag{189}$$

$$= \delta_{ik} w^{\bar{i}} \tag{190}$$

$$= w_{\bar{k}} \tag{191}$$

and similarly

$$\partial_{\bar{k}}\sigma = w_k \tag{192}$$

As well as this,

$$\partial_k A = -3c\sigma^{-4} w_{\bar{k}},\ \partial_{\bar{k}} A = -3c\sigma^{-4} w_k \tag{193}$$

Using all of this, we can show that J is closed:

$$\partial_k J_{i\bar{j}} = i[4c\sigma^{-5}A^{-\frac{2}{3}} w_i w_{\bar{j}} w_{\bar{k}} - c^2\sigma^{-4}A^{-\frac{2}{3}} w_{\bar{k}}\delta_{i\bar{j}} - 2c^2\sigma^{-8}A^{-\frac{5}{3}} w_{\bar{k}} w_i w_{\bar{j}} - c\sigma^{-4}A^{-\frac{2}{3}} w_{\bar{j}}\delta_{ik}] \tag{194}$$

and so as a (2,1)-form we have that the δ terms vanish by symmetry, so we are left with

$$\partial J = 2ic\sigma^{-5}A^{-\frac{2}{3}}(2 - c\sigma^{-3}A^{-1}) w_{\bar{k}} w_i w_{\bar{j}}\, dw^k \wedge dw^i \wedge dw^{\bar{j}} \tag{195}$$

$$= \partial_k J_{i\bar{j}}\, dw^k \wedge dw^i \wedge dw^{\bar{j}} \tag{196}$$

where $\partial_k J_{i\bar{j}}$ is symmetric in it's indices. Therefore, by Lemma 1.3 we have that

$$\partial J = 0 \tag{197}$$

where the same can be done to show $\bar{\partial} J = 0$. Therefore,

$$dJ = (\partial + \bar{\partial})J = 0 \tag{198}$$

and J is therefore closed, and we have a Kähler form $\Rightarrow b_{11} = 1$ □

Theorem 2.9 (Smoothing of $T^6/\mathbb{Z}_3$ Orbifold)**.** *Recall Example 2.3, which had 27 orbifold singularities. Around these singularities, for a small enough region, the space will look like $\mathbb{R}^6/\mathbb{Z}_3$ [17]. This is what our EH_3 looks like asymptotically from Lemma 2.7. What we want to do is to remove a very small ball from around the singularities of our orbifold and replace them with a hypersurface of the ALE EH_3, i.e. setting $\sigma = R$ for constant R [5]. This can be done by letting $c << 1, R >> 1$ such that we obtain the ALE condition quickly, and then the boundary of our patch on the orbifold matches that of the boundary of our patch of the ALE space, both of which are $S^5/\mathbb{Z}_3$ as this is the ball that we excise from both $T^6/\mathbb{Z}_3$ and EH_3 [2]. The conditions on c and σ ensure that this is done smoothly [2] [5]. The resulting manifold, with singularities smoothed out by the 27 EH_3 patches is Calabi-Yau [5]. We would like to show that this is in fact Calabi-Yau, but this is beyond the scope of this thesis. Due to Lemma 2.8, for every EH_3 patch that we add we get an additional (1,1)-form, and as there are 27 orbifold singularities to patch this means we have an additional 27 (1,1)-forms, as well as the original 9 (1,1)-forms from the $T^6/\mathbb{Z}_3$ orbifold [5]. Therefore, our resulting Calabi-Yau manifold has it's Hodge diamond given by*

$$\begin{pmatrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 36 & & 0 & \\ 1 & & 0 & & 0 & & 1 \\ & 0 & & 36 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{pmatrix} \tag{199}$$

From Theorem 2.1, we have the Euler Characteristic of this manifold being

$$\chi = 72 \tag{200}$$

which is stated in both [5] and [17].

Finally, then, we have an example of a 3-complex-dimensional simply-connected Calabi-Yau manifold so that we can write the Hodge diamond as given by Theorem 2.7. Calabi-Yau manifolds of this form are important for us in string theory reductions, as the Hodge numbers tell us information about the field content of the resulting 4D theory. As mentioned previously, we assume in our string theory reductions that the only unique Hodge numbers are b_{11}, b_{12}, and so to use an example manifold for our reductions we require that this is the case.

3 String Theory Dimensional Reduction

Now that we have equipped ourselves with an appropriate understanding of the required geometry, we can begin to consider string theory reductions. What we hope is that having additional extra dimensions, beyond just the single extra dimension of the motivating example of Kaluza-Klein Theory, will allow us to arrive at a more comprehensive effective field theory. The ultimate goal of string theory reductions would be to unify gravity with the standard model, so we would require a number of scalar fields and vector fields in the final 4-dimensional action that match the content of the standard model. Interestingly, what we will see in the following Subsection is that the Hodge Diamond of the Calabi-Yau that we use to represent the extra six dimensions ends up determining how many fields we get in the 4-dimensional action.

3.1 Dimensional Reduction of Type IIA String Theory

We can begin our reduction of Type IIA string theory by stating the action and describing it's content. There are four other types of string theory with their different field content, and there are various symmetries between them. We pick Type IIA as it has a symmetry with M-Theory that we'd like to consider later to bridge from string theory to M-Theory for Section 4.

Definition 3.1 (Type IIA Action)**.** *We define the bosonic action of Type IIA string theory as*

$$S_A = S_{NS} + S_R + S_{CS} \tag{201}$$

which correspond to the actions for what are called the NSNS and RR sectors, as well as an action referred to as the Chern-Simons term. We can write these individual actions as follows:

$$S_{NS} = \frac{1}{2\kappa^2}\int_M e^{-2\tilde{\phi}}\{\tilde{\mathscr{R}}\star 1 + 4d\tilde{\phi}\wedge\star d\tilde{\phi} - \frac{1}{2}\tilde{H}_3\wedge\star\tilde{H}_3\} \tag{202}$$

$$S_R = -\frac{1}{4\kappa^2}\int_M \{\tilde{F}_2\wedge\star\tilde{F}_2 + \tilde{F}_4\wedge\star\tilde{F}_4\} \tag{203}$$

$$S_{CS} = -\frac{1}{4\kappa^2}\int_M \{\tilde{B}_2\wedge\tilde{F}_4\wedge\tilde{F}_4\} \tag{204}$$

where the tildes represent 10-dimensional objects, and $M = \mathbb{M}_4 \times X_6$ for X_6 a Calabi-Yau manifold. In these actions, $\tilde{\mathscr{R}}$ is the Ricci-scalar obtained from the 10D metric on M,

$$\tilde{H}_3 = d\tilde{B}_2 \tag{205}$$

is the field strength of the 2-form $\tilde{B}_2$, and $\tilde{\phi}$ is a scalar field. For the other 2 terms, we have field strengths

$$\tilde{F}_2 = d\tilde{C}_1 \tag{206}$$

$$\tilde{F}_4 = d\tilde{C}_3 + \tilde{B}_2\wedge d\tilde{C}_1 \tag{207}$$

We have written this action from [2], as well as using definitions of field strengths and conventions from both [19] and [7] [5].

Our aim is to take this 10-dimensional action and use a dimensional reduction similar to Subsection 1.1 to get a 4-dimensional action. This is an involved process, more so than Subsection 1.1, and while in [7] they choose to simply display the results of this process for IIB, we will give a more detailed view for IIA using their general ideas, as well as following [13] as a guide and to confirm our results by comparing with this paper. This process combines essentially everything that has been introduced so far in the project into one computation, so while it may be lengthy, it really demonstrates how all the concepts from the previous chapters are necessary for string theory. This is an essential part of what we wish to demonstrate in this thesis - how the geometry of the extra dimensions determines the resulting 4D theory.

Theorem 3.1 (Dimensional Reduction of IIA)**.** *Due to the large size of S_A, we shall reduce each term at a time to make the calculation more manageable. To begin with S_{NS}, we say that X_6 has two independent Hodge numbers b_{11}, b_{21}, as seen for Calabi-Yau manifolds in Theorem 2.7. This is essentially assuming that X_6 is simply-connected. Our metric for M is*

$$ds^2 = g_{IJ}dx^I dx^J = g_{\mu\nu}dx^\mu dx^\nu + g_{i\bar{j}}dz^i dz^{\bar{j}} \tag{208}$$

i.e. the product metric. In our action in Definition 3.1, the Hodge star is over the whole 10-dimensional manifold M. When we make the expansions above, we now have field content either on $\mathbb{M}_4$ and X_6, and so we need to differentiate between what the Hodge star is acting on:

$$\star_{10}(A \wedge B) = (-1)^{pq} \star_4 (A) \wedge \star_6(B) \tag{209}$$

where $A \in H^p(\mathbb{M}_4), B \in H^q(X_6)$ [13]. This is just a rewrite of Lemma 1.7, just to make it clearer for this particular application. The same applies also to the exterior derivative, which has a simpler relationship: $d_{10} = d_4 + d_6$, from Lemma 1.6. As our expansions are in terms of harmonic forms on X_6, we will have that d_6 vanishes wherever it is used.

We label the basis of $H^{(1,1)}(X_6)$ as $\{\omega_i\}_{i=1}^{b_{11}}$ and make the expansions [7]

$$\tilde{\mathscr{R}} \star_{10} (1) = \mathscr{R} \star_4 (1) \wedge \star_6(1) + \frac{1}{2} dv^i(x) \wedge \omega_i(z) \wedge \star(dv^j(x) \wedge \omega_j(z)) \tag{210}$$

$$\tilde{B}_2 = B_2(x) + b^i(x)\omega_i(z) \tag{211}$$

$$\tilde{\phi} = \phi(x) \tag{212}$$

where $x \in \mathbb{M}_4,\ z \in X_6$. Here, $\phi(x), v^i, b^i$ are scalar fields on $\mathbb{M}_4$, with $i \in \{1, .., b_{11}\}$, and B_2 is a 2-form on $\mathbb{M}_4$ also. We have taken the Ricci scalar expansion from [3] and the other two from [7]. We can take expansions from [7] for our IIA reduction, even though [7] considers a IIB reduction, as both IIA and IIB share the S_{NS} term [19].

[5] Note that [7] considers the IIB action - I have simply tried to follow the style used there for our IIA action.

The Ricci scalar expansion comes from taking derivatives of the deformation of the metric on X_6:

$$\delta g_{i\bar{j}} = v^a \omega^a_{i\bar{j}} \tag{213}$$

where $\omega^a_{i\bar{j}}$ are the basis forms of $H^{(1,1)}(X_6)$, the same as in the expansion of $\tilde{B}_2$. I have chosen not to show this derivation of the expansion of $\tilde{\mathscr{R}}$ as it is a lengthy calculation and does not give us more insight than the reduction of the 5D Einstein-Hilbert action in Theorem 1.1, so considering that example just with extra dimensions will be sufficient. As $\tilde{\phi}$ is just a scalar on M, we can simply just let it depend only on $\mathbb{M}_4$ and this gives us our 4D expansion as above. This is similar to taking the zero modes in the Fourier expansion in Definition 1.2.

We would like to define the product on $H^{(1,1)}(X_6)$ via the ω forms, as this will help to simplify our expressions. We use the following: [7]

$$G_{ij} = \int_{X_6} \omega_i \wedge \star_6 \omega_j \tag{214}$$

and can then begin to substitute our expansions into S_{NS}:

$$\begin{aligned} S_{NS} = \frac{1}{2\kappa^2}\int_{\mathbb{M}_4\times X_6} e^{-2\phi}\{&\mathscr{R}\star_4(1)\wedge\star_6(1) + \frac{1}{2}(d_{10}v^i)\wedge\omega^i\wedge\star_{10}(d_{10}(v^j)\wedge\omega^j) + 4d_{10}\phi\wedge\star_{10}d_{10}\phi \\ &-\frac{1}{2}d_{10}(B_2+b^i\omega^i)\wedge\star_{10}d_{10}(B_2+b^j\omega^j)\} \end{aligned} \tag{215}$$

$$\begin{aligned} = \frac{1}{2\kappa^2}\int_{\mathbb{M}_4\times X_6} e^{-2\phi}\{&\mathscr{R}\star_4(1)\wedge\star_6(1) + \frac{1}{2}(d_4v^i)\wedge\omega_i\wedge\star_4(d_4v^j)\wedge\star_6\omega_j + 4d_4\phi\wedge\star_4 d_4\phi \\ &-\frac{1}{2}d_4B_2\wedge\star_4 dB_2\star_6 1 - \underline{\frac{1}{2}d_4B_2\wedge\star_4(d_4b^i)\wedge\star_6\omega_i} - \underline{\frac{1}{2}d_4b^i\wedge\omega_i\wedge\star_4(d_4B_2)\star_6 1} \\ &-\frac{1}{2}d_4b^i\wedge\omega_i\wedge\star_4(d_4b^j)\wedge\star_6\omega_j\} \end{aligned} \tag{216}$$

$$\begin{aligned} = \frac{1}{2\kappa^2_{(4)}}\int_{\mathbb{M}_4} e^{-2\phi}\{&\mathscr{R}\star_4 1 - \frac{1}{2}G_{ij}d_4v^i\wedge\star_4 d_4v^j + 4d_4\phi\wedge\star_4 d_4\phi \\ &-\frac{1}{2}d_4B_2\wedge\star_4 d_4B_2 + \frac{1}{2}G_{ij}d_4b^i\wedge\star_4 d_4b^j\} \end{aligned} \tag{217}$$

We see that the underlined terms in Equation (216) vanish as after the Hodge star acts on the forms we end up getting a 6-form on $\mathbb{M}_4$ and an 8-form on X_6, which both obviously vanish. This argument will be key for the rest of this reduction of IIA, and we will continue to underline terms where this happens for the rest of this thesis. The reason why we have simply replaced d_{10} for d_4 is because the forms on X_6 that we have included in our expansion are harmonic forms, and so these terms vanish under d_6, and d_6 acting on fields on $\mathbb{M}_4$ vanish due to being independent of the $\mathbb{M}_4$ coordinates. We get some sign changes in Equation (217) due to the permuting of wedge products to get the form required for G_{ij}. We will do this without mentioning it in the future as it's of little consequence overall.

Now we can begin to consider the S_R term. We take the following expansions: [13]

$$\tilde{C}_1 = C_1(x) \tag{218}$$
$$\tilde{C}_3 = C_3(x) + A_1^i(x) \wedge \omega_i(z) + p^k(x)\alpha_k(z) + q_k(x)\beta^k(z) \tag{219}$$

This expansion is essentially all possible ways to obtain a 3-form from the harmonic expansions of X_6: $\{\alpha_k\}_{k=1}^{b_{21}}$ is the basis of harmonic forms on $H^{(2,1)}$, $\{\beta^k\}_{k=1}^{b_{12}}$ is the basis of harmonic forms on $H^{(1,2)}$. Note that $b_{12} = b_{21}$ as these spaces are related by the complex conjugate of one another, and so $H^{(2,1)}, H^{(1,2)}$ are dual spaces. The term $A_1^i \wedge \omega_i$ is a 3-form obtained by the wedge with the basis of forms ω_i on $H^{(1,1)}$ and a collection of 1-forms A^i on $\mathbb{M}_4$.

From these expansions, we can begin to consider what S_R will look like. Clearly,

$$\int_{\mathbb{M}_4 \times X_6} \tilde{F}_2 \wedge \star_{10} \tilde{F}_2 = \int_{\mathbb{M}_4 \times X_6} d_{10}\tilde{C}_1 \wedge \star_{10} d_{10}\tilde{C}_1 \tag{220}$$
$$= \int_{\mathbb{M}_4} d_4 C_1 \wedge \star_4 d_4 C_1 \tag{221}$$

so we can simply say

$$\int_M \tilde{F}_2 \wedge \star_{10} \tilde{F}_2 = \int_{\mathbb{M}_4} F_2 \wedge \star_4 F_2 \tag{222}$$

The expansion of the $\tilde{F}_4$ term is more complicated:

$$\int_M \tilde{F}_4 \wedge \star \tilde{F}_4 = \int_{\mathbb{M}_4 \times X_6} \{ d_{10}\tilde{C}_3 \wedge \star_{10} d_{10}\tilde{C}_3 + \tilde{B}_2 \wedge d_{10}\tilde{C}_1 \wedge \star_{10}(\tilde{B}_2 \wedge d_{10}\tilde{C}_1) \tag{223}$$
$$- d_{10}\tilde{C}_3 \wedge \star_{10}(\tilde{B}_2 \wedge d_{10}\tilde{C}_1) - \tilde{B}_2 \wedge d_{10}\tilde{C}_1 \wedge \star_{10} d_{10}\tilde{C}_3 \}$$

$$= \int_{\mathbb{M}_4 \times X_6} \{ d_{10}\tilde{C}_3 \wedge \star_{10} d_{10}\tilde{C}_3 + \tilde{B}_2 \wedge d_{10}\tilde{C}_1 \wedge \star_{10}(\tilde{B}_2 \wedge d_{10}\tilde{C}_1) \tag{224}$$
$$- 2\tilde{B}_2 \wedge d_{10}\tilde{C}_1 \wedge \star_{10} d_{10}\tilde{C}_3 \}$$

We can combine the final two terms in Equation (223) as the exterior product of two forms is symmetric (Definition 1.11). This will be quite a long calculation to do all at once, so we'll split the expansion of these three terms up and do them separately, beginning with the first term:

$$\int_{\mathbb{M}_4\times X_6} d_{10}\tilde{C}_3\wedge\star_{10}d_{10}\tilde{C}_3$$
$$=\int_{\mathbb{M}_4\times X_6} d_{10}(C_3+A_1^i\wedge\omega_i+p^k\alpha_k+q_k\beta^k)\wedge\star_{10}d_{10}(C_3+A_1^i\wedge\omega_i+p^k\alpha_k+q_k\beta^k) \quad (225)$$

$$=\int_{\mathbb{M}_4\times X_6} d_4C_3\wedge\star_4 d_4C_3+d_4A_1^i\wedge\omega_i\wedge\star_4 d_4A_1^j\wedge\star_6\omega_j \quad (226)$$
$$+\underline{d_4A_1^i\wedge\omega_i\wedge\star_4d_4C_3\star_6 1}+\underline{d_4C_3\wedge\star_4d_4A_1^i\wedge\star_6\omega_i}+\underline{d_4C_3\wedge\star_4d_4p^k\wedge\star_6\alpha_k}$$
$$+\underline{d_4p^k\wedge\alpha_k\wedge\star_4d_4C_3\star_6 1}+\underline{d_4p^k\wedge\alpha_k\wedge\star_4dA_1^i\wedge\star_6\omega_i}+\underline{dA_1^i\wedge\omega_i\wedge\star_4d_4p^k\wedge\star_6\alpha_k}$$
$$+\underline{d_4q_k\wedge\beta^k\wedge\star_4d_4C_3\star_6 1}+\underline{d_4C_3\wedge\star_4d_4q_k\wedge\star_6\beta^k}+\underline{d_4q_k\wedge\beta^k\wedge\star_4d_4A_1^i\wedge\star_6\omega_i}$$
$$+\underline{d_4A_1^i\wedge\omega_i\wedge\star_4d_4q_k\wedge\star_6\beta^k}+d_4q_k\wedge\beta^k\wedge\star_4d_4p^l\wedge\star_6\alpha_l+d_4p^k\wedge\alpha_k\wedge\star_4d_4q_l\wedge\star_6\beta^l$$
$$+d_4q_k\wedge\beta^k\wedge\star_4d_4q_l\wedge\star_6\beta^l+d_4p^k\wedge\alpha_k\wedge\star_4d_4p^l\wedge\star_6\alpha_l$$

where we have continued to underline all terms that vanish due to containing an N-form on an n-dimensional space where $N>n$. At this point we need to make some clarifications about the relationships between the basis elements α,β:

$$\int_{X_6}\alpha_k\wedge\star\beta^l=(M_1)_k^l \quad (227)$$
$$\int_{X_6}\alpha_k\wedge\star\alpha_l=(M_2)_{kl} \quad (228)$$
$$\int_{X_6}\beta^k\wedge\star\beta^l=(M_3)^{kl} \quad (229)$$

where these three new matrices involve coupling values [13]. I'm not going to go into what these three matrices involve as I don't believe they are particularly insightful for this project. With this, then, we can write the reduction of the first term as:

$$\int_{\mathbb{M}_4\times X_6} d\tilde{C}_3\wedge\star d\tilde{C}_3=\int_{\mathbb{M}_4} dC_3\wedge\star dC_3-G_{ij}dA_1^i\wedge\star dA_1^j-(M_1)_k^l dq_k\wedge\star dp^l$$
$$-(M_1)_k^l dp^k\wedge\star d_4q_l-(M_3)^{kl}dq_k\wedge\star dq_l-(M_2)_{kl}dp^k\wedge\star dp^l \quad (230)$$

To then continue with the second term of the $\tilde{F}_4$ reduction:

$$\int_{\mathbb{M}_4\times X_6}\tilde{B}_2\wedge d_{10}\tilde{C}_1\wedge\star_{10}(\tilde{B}_2\wedge d_{10}\tilde{C}_1)$$
$$=\int_{\mathbb{M}_4\times X_6}(B_2+b^i\omega_i)\wedge d_4C_1\wedge\star_{10}((B_2+b^j\omega_j)\wedge d_4C_1) \quad (231)$$

$$=\int_{\mathbb{M}_4\times X_6} B_2\wedge d_4C_1\wedge\star_4(B_2\wedge d_4C_1)\wedge\star_6 1+\underline{B_2\wedge d_4C_1\wedge\star_6(b^j\omega_j)\wedge\star_4d_4C_1} \quad (232)$$
$$+\underline{b^i\omega_i\wedge d_4C_1\wedge\star_4(B_2\wedge d_4C_1)\wedge\star_6 1}+b^i\omega_i\wedge d_4C_1\wedge\star_6(b^j\omega_j)\wedge\star_4d_4C_1$$

$$=\int_{\mathbb{M}_4} B_2\wedge dC_1\wedge\star(B_2\wedge dC_1)-G_{ij}b^idC_1\wedge\star(b^jdC_1) \quad (233)$$

where again I have underlined terms that vanish due to the dimension of the forms included.

Finally, the third term of the $\tilde{F}_4$ reduction:

$$\int_{\mathbb{M}_4\times X_6} \tilde{B}_2 \wedge d_{10}\tilde{C}_1 \wedge \star_{10} d_{10}\tilde{C}_3$$
$$= \int_{\mathbb{M}_4\times X_6} (B_2 + b^i\omega_i) \wedge d_4C_1 \wedge \star_{10}(d_4C_3 + d_4A_1^j \wedge \omega_j + d_4p^k \wedge \alpha_k + d_4q_k \wedge \beta^k) \tag{234}$$

$$\begin{aligned}
&= \int_{\mathbb{M}_4\times X_6} B_2 \wedge d_4C_1 \wedge \star_4 d_4C_3 \wedge \star_6 1 + \underline{B_2 \wedge d_4C_1 \wedge \star_4 d_4A_1^j \wedge \star_6 \omega_j} \\
&- \underline{B_2 \wedge d_4C_1 \wedge \star_4 d_4p^k \wedge \star_6 \alpha_k} - \underline{B_2 \wedge d_4C_1 \wedge \star_4 d_4q_k \wedge \star_6 \beta^k} \\
&+ \underline{b^i\omega_i \wedge d_4C_1 \wedge \star_4 d_4C_3} + b^i\omega_i \wedge d_4C_1 \wedge \star_4 d_4A_1^j \wedge \star_6 \omega_j \\
&- \underline{b^i\omega_i \wedge d_4C_1 \wedge \star_4 d_4p^k \wedge \star_6 \alpha_k} - \underline{b^i\omega_i \wedge d_4C_1 \wedge \star_4 d_4q_k \wedge \star_6 \beta^k}
\end{aligned} \tag{235}$$

$$= \int_{\mathbb{M}_4} B_2 \wedge dC_1 \wedge \star dC_3 + G_{ij} b^i dC_1 \wedge \star dA_1^j \tag{236}$$

We can now bring all of the terms for the $\tilde{F}_4$ reduction together to say

$$\begin{aligned}
&\int_{\mathbb{M}_4\times X_6} \tilde{F}_4 \wedge \star\tilde{F}_4 \qquad (237)\\
&= \int_{\mathbb{M}_4} \{dC_3 \wedge \star dC_3 - G_{ij} dA_1^i \wedge \star dA_1^j - (M_1)_k^l dq_k \wedge \star dp^l \\
&- (M_1)_k^l dp^k \wedge \star d_4q_l - (M_3)^{kl} dq_k \wedge \star dq_l - (M_2)_{kl} dp^k \wedge \star dp^l \\
&+ B_2 \wedge dC_1 \wedge \star(B_2 \wedge dC_1) - G_{ij} b^i dC_1 \wedge \star(b^j dC_1) - 2B_2 \wedge dC_1 \wedge \star dC_3 - 2G_{ij} b^i dC_1 \wedge \star dA_1^j\}
\end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{M}_4} \{dC_3 \wedge \star dC_3 - 2(B_2 \wedge dC_1) \wedge \star dC_3 + (B_2 \wedge dC_1) \wedge \star(B_2 \wedge dC_1) \\
&- G_{ij}(dA_1^i \wedge \star dA_1^j + 2b^i dC_1 \wedge \star dA_1^j + b^i dC_1 \wedge \star(b^j dC_1)) \\
&- (M_1)_k^l (dp^k \wedge \star dq_l + dq_l \wedge \star dp^k) - (M_2)_{kl} dp^k \wedge \star dp^l - (M_3)^{kl} dq_k \wedge \star dq_l\}
\end{aligned} \tag{238}$$

which is our final expression for the $\tilde{F}_4$ reduction. Thankfully, this result agrees with [13].

The only remaining thing to do is to reduce the Chern-Simons term:

$$\begin{aligned}
&\int_{\mathbb{M}_4\times X_6} \{\tilde{B}_2 \wedge \tilde{F}_4 \wedge \tilde{F}_4\} \\
&= \int_{\mathbb{M}_4\times X_6} \{(B_2 + b^i\omega_i) \wedge (d_4C_3 + d_4A_1^j \wedge \omega_j + d_4p^k \wedge \alpha_k + d_4q_k \wedge \beta^k + B_2 \wedge d_4C_1 + b^j\omega_j \wedge d_4C_1) \\
&\quad \wedge (d_4C_3 + d_4A_1^m \wedge \omega_m + d_4p^l \wedge \alpha_l + d_4q_l \wedge \beta^l + B_2 \wedge d_4C_1 + b^m\omega_m \wedge d_4C_1)\}
\end{aligned} \tag{239}$$

The previous expanded terms of the IIA action have been lengthy, but this Chern-Simons term will have 72 terms in total, most of which will vanish. It would not be feasible to go through each of these terms individually, nor would it be insightful. So, for the Chern-Simons we will use some dimensional analysis of the terms involved to determine which terms will survive.

Form	Dim on $\mathbb{M}_4$ (M)	Dim on X_6 (X)
B_2	2M	0
$b^i\omega_i$	0	2X
d_4C_3	4M	0
$d_4A_1^j \wedge \omega_j$	2M	2X
$d_4p^k \wedge \alpha_k$	1M	3X
$d_4q_k \wedge \beta^k$	1M	3X
$B_2 \wedge d_4C_1$	4M	0
$b^j\omega_j \wedge d_4C_1$	2M	2X

Table 1: Dimension of forms in Chern-Simons term of IIA action

We want to have terms that have 4 indices on $\mathbb{M}_4$ and 6 indices on X_6, i.e. terms whose dimension is (4M + 6X), where the M denotes how many indices there are on $\mathbb{M}_4$ and the same for the X with X_6. Table 1 shows how many indices each term has on each manifold, and from this we can work out that the terms which survive are those of the form, where we pick one of the first two forms and two of the latter forms:

$$(2M) + (1M + 3X) + (1M + 3X) \to 4 \textit{ terms} \tag{240}$$

$$(2X) + (2M + 2X) + (2M + 2X) \to 4 \textit{ terms} \tag{241}$$

where the order of the terms in these dimensional sums corresponds to the choice of form from each of the three wedges in Equation (239). Thus, we can choose these surviving 8 terms as follows:

$$\int_{\mathbb{M}_4 \times X_6} \{B_2 \wedge (d_4p^k \wedge \alpha_k + d_4q_k \wedge \beta^k) \wedge (d_4p^l \wedge \alpha_l + d_4q_l \wedge \beta^l) \tag{242}$$
$$+ b^i\omega_i \wedge (d_4A_1^j \wedge \omega_j + b^j\omega_j \wedge d_4C_1) \wedge (d_4A_1^m \wedge \omega_m + b^m\omega_m \wedge d_4C_1)\}$$

It's worth noting again that the Chern-Simons term is topological, and thus does not contain any Hodge stars. Because of this, we need to clarify what the wedge product of the basis elements $\alpha_k, \beta^l, \omega_i$ are: [13]

$$\int_{X_6} \alpha_k \wedge \beta^l = \delta_k^l = -\int_{X_6} \beta^l \wedge \alpha_k \tag{243}$$

$$\int_{X_6} \alpha_k \wedge \alpha_l = \int_{X_6} \beta^k \wedge \beta^l = 0 \tag{244}$$

$$K_{ijm} = \int_{X_6} \omega_i \wedge \omega_j \wedge \omega_m \tag{245}$$

where Equation (243) is because α, β are dual bases to one another, Equation (244) is because these are products of basis forms, and Equation (245) are intersection numbers from Definition 1.13.

Finally, we can write out what the dimensional reduction of the Chern-Simons term is by continuing from Equation (242):

$$\int_{\mathbb{M}_4\times X_6}\tilde{B}_2\wedge\tilde{F}_4\wedge\tilde{F}_4$$
$$=\int_{\mathbb{M}_4\times X_6}\{B_2\wedge(d_4p^k\wedge\alpha_k+d_4q_k\wedge\beta^k)\wedge(d_4p^l\wedge\alpha_l+d_4q_l\wedge\beta^l) \tag{246}$$
$$+b^i\omega_i\wedge(d_4A_1^j\wedge\omega_j+b^j\omega_j\wedge d_4C_1)\wedge(d_4A_1^m\wedge\omega_m+b^m\omega_m\wedge d_4C_1)\}$$

$$=\int_{\mathbb{M}_4\times X_6}\{B_2\wedge(\underline{d_4p^k\wedge\alpha_k\wedge d_4p^l\wedge\alpha_l}+d_4p^k\wedge\alpha_k\wedge d_4q_l\wedge\beta^l \tag{247}$$
$$+d_4q_k\wedge\beta^k\wedge d_4p^l\wedge\alpha_l+\underline{d_4q_k\wedge\beta^k\wedge d_4q_l\wedge\beta^l})\}$$
$$+\int_{\mathbb{M}_4}K_{ijm}b^i(d_4A_1^j-b^jd_4C_1)\wedge(d_4A_1^m-b^md_4C_1)$$

$$=\int_{\mathbb{M}_4}\{B_2\wedge(dq_k\wedge dp^k-dp^k\wedge dq_k) \tag{248}$$
$$+K_{ijm}b^i(d_4A_1^j-b^jd_4C_1)\wedge(d_4A_1^m-b^md_4C_1)\}$$

which agrees with [13].

We have therefore managed to dimensionally reduce all three terms of the Type IIA action, and we can write down our final 4D action:

$$S_4=\frac{1}{2\kappa_{(4)}^2}\int_{\mathbb{M}_4}e^{-2\phi}\{\mathscr{R}\star 1+\frac{1}{2}G_{ij}dv^i\wedge\star dv^j+4d\phi\wedge\star d\phi \tag{249}$$
$$-\frac{1}{2}dB_2\wedge\star dB_2-\frac{1}{2}G_{ij}db^i\wedge\star db^j\}-\frac{1}{2}\{dC_1\wedge\star dC_1+dC_3\wedge\star dC_3$$
$$-2(B_2\wedge dC_1)\wedge\star dC_3+(B_2\wedge dC_1)\wedge\star(B_2\wedge dC_1)$$
$$+G_{ij}(dA_1^i\wedge\star dA_1^j-2b^idC_1\wedge\star dA_1^j+b^idC_1\wedge\star(b^jdC_1))$$
$$+(M_1)_k^l(dp^k\wedge\star dq_l+dq_l\wedge\star dp^k)+(M_2)_{kl}dp^k\wedge\star dp^l+(M_3)^{kl}dq_k\wedge\star dq_l\}$$
$$-\frac{1}{2}\{B_2\wedge(dq_k\wedge dp^k-dp^k\wedge dq_k)+K_{ijm}b^i(dA_1^j-b^jdC_1)\wedge(dA_1^m-b^mdC_1)\}$$

where we have now got a theory with 4D field content:

- *Ricci Scalar* $\mathscr{R}$
- $2b_{11}+2b_{12}+1$ *scalar fields:* v^i,b^i,p^k,q_k,ϕ
- $b_{11}+1$ *1-forms:* C_1,A_1^i
- *A 2-form:* B_2
- *A 3-form:* C_3

with various coupling terms of these fields. In [7], [13] some of the scalar fields are combined together to create complex scalar fields, but we won't do this.

To ensure that this reduction is consistent, we should also check that the equations of motions are the same from the two actions. We can do this via variational calculus. We will not do this for the IIA reduction as this will be a significantly large calculation due to the number of fields in the 4D action, but we have done this later in Theorem 4.3 as this is simpler.

In Theorem 3.1 we had X_6 being a general Calabi-Yau manifold. We can consider an example from Subsection 2.3 and use that as our extra 6-dimensional space in the reduction. Really, the only thing to consider are the Hodge numbers, as these tell us what the resulting field content will be.

Example 3.1 (Smoothed $T^6/\mathbb{Z}_3$ Reduction)**.** *The example that we will choose is the $T^6/\mathbb{Z}_3$ with it's orbifold singularities smoothed by Eguchi-Hanson spaces, namely the manifold from Theorem 2.9. The Hodge numbers of this manifold were given as*

$$b_{11} = 36, \;\; b_{12} = 0 \tag{250}$$

This then means that our resulting field content in 4D is:

- *Ricci scalar $\mathscr{R}$*
- *73 scalar fields*
- *37 1-forms*
- *A 2-form*
- *A 3-form*

Consider the standard model - by counting each field in the action individually we see it has a grand total of 61 fields. Counting the number of fields we have (not including the Ricci scalar), we get 102. This means that this example manifold is likely not the manifold that represents the possible extra dimensions of our universe. Moreover, we have considered only the bosonic IIA action, and the standard model contains both bosons and fermions. The idea is that by choosing a correct Calabi-Yau, we would end up with the same field content as the standard model, as well as a Ricci scalar to describe general relativity. Sadly, choosing the correct manifold is a very difficult problem. There is a problem in physics called The Swampland which addresses this - we'll speak more about it in our conclusions in Section 5.

3.2 M-Theory and IIA

We have seen in the previous subsection that we can dimensionally reduce Type IIA string theory into an effective 4D action. Something we'd like to consider before the next section is how we can achieve the IIA action from an 11-dimensional action. This 11D action is the low-energy limit of M-Theory, known as 11D Supergravity. We can state the action as follows:

Definition 3.2 (Action of 11D Supergravity)**.** *We have an action over an 11-dimensional manifold M_{11} given by*

$$S = \frac{1}{2\kappa_{(11)}^2} \int_{M_{11}} \tilde{R} \star_{11} 1 - \frac{1}{2} \tilde{F}_4 \wedge \star \tilde{F}_4 - \frac{1}{6} \tilde{A}_3 \wedge \tilde{F}_4 \wedge \tilde{F}_4 \tag{251}$$

where $\tilde{F}_4 = d\tilde{A}_3$ [19].

We can see that this action has spectrum $\tilde{g}_{IJ}, \tilde{A}_3$. In the following theorem, we will find a duality between this low-energy M-Theory and Type IIA string theory, which has spectrum $g_{IJ}, B_2, \phi, C_3, C_1$.

Theorem 3.2 (M-Theory on $S^1 \cong$ IIA)**.** *For 11-dimensional supergravity given in Definition 3.2 on a manifold $M_{11} = M_{10} \times S^1$, by dimensional reduction of the S^1 we arrive at the same action for Type IIA string theory given in Definition 3.1. That is, 11-dimensional supergravity on $M_{10} \times S^1 \cong$ Type IIA string theory [19].*

Proof:

We begin with the action above in Definition 3.2, and say that our 11-dimensional manifold $M_{11} = M_{10} \times S^1$. Think back to Subsection 1.1 where we made the Kaluza-Klein ansatz in Definition 1.2 to get a vector and a scalar field from the 5D metric - we will do the same here in 11D to 10D:

$$\tilde{g}_{IJ} = \begin{pmatrix} g_{\mu\nu} & C_\nu \\ C_\mu & \phi \end{pmatrix} \tag{252}$$

where $I, J = 0, .., 10$ and $\mu, \nu = 0, .., 9$. C_μ then gives us a 1-form which we'll refer to as C_1, and we also get a scalar field ϕ. We do a similar process for $\tilde{A}_3$ by splitting $\tilde{A}_{IJK}$ into: [19]

$$\tilde{A}_{\mu\nu\lambda} = C_{\mu\nu\lambda} \tag{253}$$

$$\tilde{A}_{\mu\nu 10} = B_{\mu\nu} \tag{254}$$

where we refer to the corresponding 3-form and 2-form as C_3, B_2, respectively.

To reduce the action into 10D, we first consider Theorem 1.1 and realise that we essentially have the exact same scenario for the Ricci scalar term, just in 11D to 10D here. Using the same form for the Ricci Scalar as the one given in [12], and using the following expansions from 11D into 10D, we write, where x^{10} is the coordinate of the S^1:

$$\tilde{R} \star_{11} 1 = [R \star_{10} 1 - \frac{1}{4} F_2 \wedge \star_{10} F_2 + 4 d\phi \wedge \star_{10} \phi] \wedge dx^{10} \tag{255}$$

$$\tilde{F}_4 \wedge \star_{11} \tilde{F}_4 = d\tilde{A}_3 \wedge \star_{11} d\tilde{A}_3 = [dC_3 \wedge \star_{10} dC_3 + dB_2 \wedge \star_{10} dB_2] \wedge dx^{10} \tag{256}$$

$$\tilde{A}_3 \wedge d\tilde{A}_3 \wedge d\tilde{A}_3 = B_2 \wedge dC_3 \wedge dC_3 \wedge dx^{10} \tag{257}$$

where we have

$$F_2 = dC_1 \tag{258}$$

$$F_4 = dC_3 \tag{259}$$

and $\star_{10}$ is the Hodge star on M_{10}. Here we write d as the exterior derivative on M_{10} as we choose that nothing in our expansion, other than dx^{10}, depends on the S^1. Note that our field strength of C_3 in Equation (259) does not quite match the form given in Equation (207), but we could choose to just rescale F_4 here to match this[6]*.*

In Equation (255) we have, as mentioned, used a previously used expansion, however we have rescaled the ϕ term to obtain a preferable constant for later on. As well as this, we've turned the $F_{\mu\nu}F^{\mu\nu}$ into 2-form notation, as well as the $\partial_\mu\phi\partial^\mu\phi$ term. In Equation (256), we used the fact that $\tilde{A}_3$ can be decomposed into a 3-form and 2-form in 10D as seen in Equations (253) and (254), and the summation over the indices gives us this form. Equation (257) is due to the fact that this is the only expansion of the three $\tilde{A}_3$ in this term that gives a 10-form on M_{10} - this is the same dimensional analysis approach to what we did for the Chern-Simons term in the IIA reduction.

Therefore, we can write our 11D action as

$$\begin{aligned} S = \frac{1}{2\kappa^2_{(11)}} \int_{M_{10}\times S^1} \{ & \mathscr{R} \star_{10} 1 - \frac{1}{4} F_2 \wedge \star_{10} F_2 + 4 d\phi \wedge \star_{10} d\phi \\ & - \frac{1}{2}(dC_3 \wedge \star_{10} dC_3 + dB_2 \wedge \star_{10} dB_2) \\ & - \frac{1}{6}(B_2 \wedge dA_3 \wedge dA_3)\} \wedge dx^{10} \end{aligned} \tag{260}$$

$$\begin{aligned} = \frac{\pi R}{\kappa^2_{(11)}} \int_{M_{10}} \{ & \mathscr{R} \star 1 - \frac{1}{4} F_2 \wedge \star F_2 + 4 d\phi \wedge \star d\phi \\ & - \frac{1}{2}(F_4 \wedge \star F_4 + H_3 \wedge \star H_3) \\ & - \frac{1}{6}(B_2 \wedge F_4 \wedge F_4)\} \end{aligned} \tag{261}$$

where we have field strengths defined as $F_k = dC_{k-1}, H_3 = dB_2$. Comparing this result with the action in Definition 3.1 we can see that, ignoring some scaling of various fields, this is the Type IIA action. We can compare the coupling constant from our 11-dimensional action with the IIA action:

$$\kappa^2 = \frac{\kappa^2_{(11)}}{2\pi R} \tag{262}$$

so we see that as the radius of the extra dimension scales, the strength of the coupling of the IIA action is scaled inversely. Thus, the size of the 11th dimension determines the strength of the 10-dimensional theory.

[6] This mismatch is not addressed in [19].

4 M-Theory Dimensional Reduction

In the previous chapter we reduced Type IIA string theory on 6-dimensional Calabi-Yau manifolds, and saw that we obtained a 4-dimensional theory with some interesting field content. Additionally, we saw that the IIA action can be derived from an 11-dimensional low-energy limit to M-theory. In that derivation, we said that the extra 7-dimensional manifold was a sum of a Calabi-Yau and a circle, and so the IIA reduction in Theorem 3.1 could be considered, by taking Theorem 3.2 into account, as a reduction of M-theory on this 7-dimensional $X_6 \times S^1$. The purpose of this chapter is to introduce M-theory reductions on more interesting 7-dimensional manifolds, namely G_2 manifolds. In the first subsection we will introduce the geometry of these manifolds and then in the second subsection we will dimensionally reduce the action in Definition 3.2 on G_2 manifolds. The gauge symmetry group of the standard model is $SU(3) \times SU(2) \times U(1)$, which to be represented as a Lie group requires 7 dimensions [22], and so to have this appear as the resulting gauge symmetry for a 4D theory we require 11 dimensions. The action we saw in Definition 3.2 was for 11D supergravity, which is a low energy approximation of M-theory. A fascinating coincidence is that 11 dimensions is the maximum dimension for supergravity [22], and the minimum required for the standard model gauge symmetry. This could be an indicator that M-theory could be the correct theory to unify gravity with the standard model. With this, we shall begin our study of G_2-manifolds, the required extra dimensions for M-theory. We will then dimensionally reduce M-theory on G_2-manifolds, analyse the 4D field content and then see whether the 4D theory has the same gauge symmetry as the standard model.

4.1 G2 Manifolds

For our discussion of G_2 manifolds, we turn to [10] as a main reference. Additionally, we use parts of [2] where appropriate. An interesting thing to note is that G_2 manifolds are sometimes referred to as Joyce manifolds, as Joyce was the first to construct compact manifolds with holonomy G_2 [9]. These are relatively new manifolds and are still being actively researched.

Definition 4.1 (Exceptional Lie group G_2). *Let $x_I \in \mathbb{R}^7$ such that we can define a 3-form on $\mathbb{R}^7$ by*

$$\varphi_0 = dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{347} - dx^{356} \tag{263}$$

where we introduce notation $dx^{ijk} = dx^i \wedge dx^j \wedge dx^k$ [10]. We define the Lie group G_2 as the group of transformations that leave φ_0 invariant, i.e. by using the definition of a stabilizer group from Definition 2.8 we say that [10]

$$G_2 = stab(\varphi_0) \tag{264}$$

We can use the induced Euclidean metric from $\mathbb{R}^7$ to take the Hodge star of φ_0 to get a 4-form [10]

$$\star\varphi_0 = dx^{4567} + dx^{2367} + dx^{2345} + dx^{1357} - dx^{1346} - dx^{1256} - dx^{1247} \tag{265}$$

where each term from φ has simply had it's choice of indices inverted.

Definition 4.2 (G_2-Manifolds). *Our definition of a G_2-manifold is a 7-dim compact, simply-connected manifold (M,g) that has an isomorphism $T_pM \cong \mathbb{R}^7$ such that we have a 3-form φ and 4-form $\star\varphi$ on T_pM that are isomorphic to $\varphi_0, \star\varphi_0$ on $\mathbb{R}^7$ such that [10]*

$$d\varphi = 0 \tag{266}$$

$$d \star \varphi = 0 \tag{267}$$

It can be shown that this condition is equivalent to M having holonomy G_2, but we will usually use the former definition for a G_2-Manifold.

Corollary 4.1 (Harmonic 3-form). *The closed 3-form φ that defines a G_2-manifold is harmonic.*
Proof: From Equation (267) we can apply a Hodge star on the left-hand side to get:

$$\star d \star \varphi = 0 \tag{268}$$

Recall Definition 1.14, and notice that this is what we have above, so we can write

$$d^{\dagger}\varphi = 0 \tag{269}$$

which means that the Laplacian acting on φ is

$$\triangle\varphi = (dd^{\dagger} + d^{\dagger}d)\varphi = 0 \quad \square \tag{270}$$

Theorem 4.1 (Calabi-Yau $\times$ S^1 is G_2). *Let (X, h) be a simply connected Calabi-Yau with real $\dim X = 6$, with Kähler form J and holomorphic (3,0)-form Ω. Let $\theta \in S^1$ such that S^1 has a 1-form $d\theta$. Then*

$$\varphi = d\theta \wedge J + Re(\Omega) \tag{271}$$

$$\star\varphi = J \wedge J - d\theta \wedge Im(\Omega) \tag{272}$$

define a G_2-Manifold [7] *[10].*

Proof: A proof of this is not included in [10], so we shall do this now. All we need to do is show that $\varphi, \star\varphi$ are both closed. Beginning with φ,

$$d\varphi = d(d\theta \wedge J) + d(Re(\Omega)) \tag{273}$$

$$= d(d\theta) \wedge J - d\theta \wedge dJ \tag{274}$$

$$= 0 \tag{275}$$

where $d(Re(\Omega))$ vanishes due to Ω being harmonic and dJ vanishes as X is Calabi-Yau and thus Kähler, so the Kähler form J must be closed. $d\theta$ is closed as it provides a basis for $H^1(S^1)$ and is therefore harmonic.

We can show how to derive $\star\varphi$ from φ, and then show it's also closed:

[7] [10] uses a normalisation constant of $\frac{1}{2}$ for the $J \wedge J$ term, which we do not include.

$$\star\varphi = (-1)^{np} \star_7 (d\theta \wedge J + 1 \wedge Re(\Omega)) \tag{276}$$
$$= - \star_7 (d\theta \wedge J) - \star_7 (1 \wedge Re(\Omega)) \tag{277}$$
$$= + \star_1 (d\theta) \wedge \star_6 (J) - \star_1 (1) \wedge \star_6 (Re(\Omega)) \tag{278}$$
$$= 1 \wedge J \wedge J - d\theta \wedge Im(\Omega) \tag{279}$$
$$= J \wedge J - d\theta \wedge Im(\Omega) \tag{280}$$

where $\star_1$ is the Hodge star on S^1 and $\star_6$ is the Hodge star on X. We've used that the Hodge star of the Kähler form is the Kähler form wedged with itself, as well as the Hodge star of the real part of the (3,0)-form being the imaginary part of the (3,0)-form [8]. We can show that this is closed:

$$d \star \varphi = d(J \wedge J) - d(d\theta \wedge Im(\Omega)) \tag{281}$$
$$= dJ \wedge J + J \wedge dJ - d(d\theta) \wedge Im(\Omega) + d\theta \wedge d(Im(\Omega)) \tag{282}$$
$$= 0 \quad \square \tag{283}$$

Lemma 4.1. *The Betti numbers of a G_2-Manifold with $hol(g) = G_2$ are given by: [16] [10]*

$$b_0 = b_7 = 1, \;\; b_1 = b_6 = 0, \;\; b_2 = b_5, \;\; b_3 = b_4 \tag{284}$$

Note that if $hol(g) \subset G_2$ then b_1 does not necessarily vanish. We see that this means the Betti numbers are completely free in the case of $hol(g) \subset G_2$.

Now that we have introduced G_2-manifolds, we would like to construct some examples. We will give one example with $hol(g) \subset G_2$ through a Calabi-Yau construction, and then we'll give an example that has $hol(g) = G_2$ which does not use Calabi-Yau constructions. Our M-theory reductions assume $hol(g) = G_2$, so the latter example would be the more appropriate example to consider as the extra dimensions of an M-theory reduction.

Example 4.1 (G_2-manifold from smoothed $T^6/\mathbb{Z}_3$ orbifold)**.** *In Example 2.9 we used Eguchi-Hanson spaces to smooth the $T^6/\mathbb{Z}_3$ orbifold in Example 2.3. We said that the resulting manifold M was a 3-complex-dimensional Calabi-Yau manifold. Also, the resulting manifold was simply-connected. Therefore, we can use Theorem 4.1 and Example 2.9 to construct a smooth G_2-manifold:*

$$Y = M \times S^1 \text{ is a smooth } G_2\text{-manifold} \tag{285}$$

The Betti numbers of this G_2-manifold Y can be computed by using the Hodge numbers from Example 2.9 and the fact that [5]

$$b_0(S^1) = 1, b_1(S^1) = 1 \tag{286}$$

To obtain the Betti numbers of M from it's Hodge numbers, we use Lemma 2.4:

$$b_0(M) = 1 \tag{287}$$
$$b_1(M) = 0 \tag{288}$$
$$b_2(M) = 36 \tag{289}$$
$$b_3(M) = 2 \tag{290}$$

Using Definition 1.19, we can find that:

$$b_0(Y) = b_0(M) \cdot b_0(S^1) = 1 \tag{291}$$
$$b_1(Y) = b_0(M) \cdot b_1(S^1) = 1 \tag{292}$$
$$b_2(Y) = b_2(M) \cdot b_0(S^1) = 36 \tag{293}$$
$$b_3(Y) = b_2(M) \cdot b_1(S^1) + b_3(M) \cdot b_0(S^1) = 38 \tag{294}$$

so that finally we have Betti numbers of Y:

$$(b_0, b_1, b_2, b_3) = (1, 1, 36, 38) \tag{295}$$

Note that the reason b_1 does not vanish here, as mentioned in Lemma 4.1, is because Y has it's holonomy as a subgroup of G_2, and is not equivalent to G_2.

Example 4.2 (7-Torus Orbifold)**.** *We saw from Example 2.1 that the 6-Torus is a Calabi-Yau manifold. We can write the 6-Torus as*

$$T^6 = S^1 \times .. \times S^1 \tag{296}$$

i.e. the product of 6 circles. Thus, when considering Theorem 4.1 we might say that

$$T^6 \times S^1 = S^1 \times ... \times S^1 \times S^1 = T^7 \text{ is a } G_2\text{-manifold.} \tag{297}$$

However, the torus is not simply connected, and so we cannot say this. To construct a simply-connected G_2-manifold, we can do something similar to what we did in Example 2.3 to make our T^7 simply-connected. If we let a quotient group act on T^7 and then resolve the singularities as we did in Subsection 2.4, then we get a simply-connected, smooth G_2-manifold [10]. We consider the T^7 orbifold produced by acting on T^7 with the quotient group $\Lambda = (\mathbb{Z}_2)^3$ given in [10], i.e. we act with $\mathbb{Z}_2$ three times:

$$(x^1, .., x^7) \overset{g_1}{\approx} (x^1, x^2, x^3, -x^4, -x^5, -x^6, -x^7) \tag{298}$$
$$\overset{g_2}{\approx} (x^1, -x^2, -x^3, x^4, x^5, \frac{1}{2} - x^6, -x^7) \tag{299}$$
$$\overset{g_3}{\approx} (-x^1, x^2, -x^3, x^4, \frac{1}{2} - x^5, x^6, \frac{1}{2} - x^7) \tag{300}$$

where $g_i \in \Lambda$. We can check that these identifications leave φ, $\star\varphi$ invariant. For example, consider acting with each g_i on the term dx^{145} of φ:

$$dx^{145} \xrightarrow{g_1} dx^1 \wedge -dx^4 \wedge -dx^5 = dx^{145} \quad (301)$$

$$\xrightarrow{g_2} dx^1 \wedge dx^4 \wedge dx^5 = dx^{145} \quad (302)$$

$$\xrightarrow{g_3} -dx^1 \wedge dx^4 \wedge d(\frac{1}{2} - x^5) \quad (303)$$

$$= -dx^1 \wedge dx^4 \wedge -dx^5 = dx^{145} \quad (304)$$

so we see that each g_i leaves this term invariant. We also have this for all the other terms of φ and $\star\varphi$. Note that these specific quotients have been chosen for the particular definition of the 3- and 4-forms φ, $\star\varphi$ from Definition 4.1. In [19] they choose a different definition, and so the quotients here are different to the quotients that [19] use for this example. If we used the quotients from [19] with the definition of the 3- and 4-forms from [10] then we would not get this symmetry. From these identifications, we can deduce the Betti numbers of our T^7/Λ orbifold by considering which forms are left invariant by the quotient group, as usual. From [19] we have that no 1-forms or 2-forms are preserved under g_i transformations, but we get 7 preserved 3-forms. These will be each of the 7 dx^{ijk} terms in φ. The 1-forms and 2-forms will vanish as there are no dx^i or $dx^i \wedge dx^j$ left unchanged by all 3 of the $\mathbb{Z}_2$ quotients in Λ. For example, the 2-form $dx^1 \wedge dx^2$ is invariant under g_1, g_2, but not g_3. Therefore the Betti numbers of our T^7/Λ orbifold are given as

$$(b_0, b_1, b_2, b_3) = (1, 0, 0, 7) \quad (305)$$

From [10] we know that T^7/Λ has 12 orbifold singularities - we might think that each g_i produces 16 singularities and thus Λ produces 48 singularities, but when we act with two or more g_i and utilise the fact that $x_k \sim x_k + 1$ by the identifications that define the 7-Torus, we end up seeing that each g_i ends up contributing only 4 singularities each, leaving us 12 in total [19].

To smooth these singularities, we should notice that the singularities mentioned above leave 3 of our coordinates unchanged, i.e., we are left with a $T^3 \times Y$ where Y contains our singularities [10] [19]. We have that locally around the singularities of Y it looks like $\mathbb{R}^4/\mathbb{Z}_2 \cong \mathbb{C}^2/\mathbb{Z}_2$ [10]. This is because locally the quotient group is $\mathbb{Z}_2$, not Λ. Thinking back to Subsection 2.4, we used the Eguchi-Hanson space EH_3 to patch our singularities around a locally $\mathbb{R}^6/\mathbb{Z}_3$ space (in Theorem 2.9). What we can do to smooth our orbifolds now is to use EH_2, which locally will look like $\mathbb{R}^4/\mathbb{Z}_2$ by the same reasoning as in Theorem 2.9. While in Lemma 2.8 we considered specifically the cohomology of EH_3, the cohomology of EH_2 takes essentially the exact same form:

$$\begin{pmatrix} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{pmatrix} \quad (306)$$

i.e., the only harmonic form is the Kähler form defined by the Eguchi-Hanson metric. This can be seen from the fact that our calculations in Lemma 2.8 did not depend on the dimension of the space, up to constants. To write our Hodge numbers as Betti numbers for EH_2, we have simply

$$(b_0, b_1, b_2) = (1, 0, 1) \tag{307}$$

By replacing the singularities of Y with EH_2 as in Theorem 2.9, we will obtain 12 copies of EH_2. This means that we will receive an additional 12 harmonic 2-forms which contributes to b_2 of our final smoothed orbifold. Adding these forms to the cohomology of the T^7/Λ orbifold, we get

$$(b_0, b_1, b_2, b_3) = (1, 0, 12, 7) \tag{308}$$

for our smoothed T^7/Λ orbifold. We know from [19], [10] that the holonomy of this final manifold is strictly G_2. The proof of this can be found in [10], but requires advanced analysis beyond the scope of this thesis. Therefore we have found an example of a smooth 'proper' G_2-manifold, i.e. a manifold with $hol(g) = G_2$.

In general, constructing 'proper' G_2-manifolds is a very difficult problem, and this provides a significant hurdle for progress in M-theory compactifications.

4.2 M-Theory on G2 Manifolds

Recall Definition 3.2, as this will be the action that we are considering in this chapter. As mentioned, this is actually the 11D supergravity action, which is a low energy approximation to M-theory.

Corollary 4.2. *Consider Theorem 3.2, where we had the 11D supergravity action on $M_{11} = M_{10} \times S^1$. Let*

$$M_{10} = \mathbb{M}_4 \times X_6 \tag{309}$$

and then appreciate that this means we have

$$M_{11} = \mathbb{M}_4 \times X_6 \times S^1 \tag{310}$$

We can then use Theorem 4.1 to say that

$$G = X_6 \times S^1 \tag{311}$$

is a G_2 manifold. This then means that by the reduction from 11D supergravity to the IIA action in Theorem 3.2 and the reduction from IIA to a 4D theory in Theorem 3.1, we have already managed to do a dimensional reduction of 11D supergravity on G to a 4D theory without even meaning to! As G is just a special case of G_2 manifold, we have only considered one case of M-theory reduction, so we would like to consider more general reductions.

This Corollary is an important reason for why we chose to do IIA reductions as opposed to one of the other 4 string theories. While we could then use other string theory dualities to bridge to the other types from this point, this one is a more natural connection between M-theory and string theory. It also shows us that string theory would not necessarily be the most fundamental theory, but a limiting case of M-theory. This is not surprising, as we would not have been able to achieve the standard model's gauge symmetry from a 10D string theory due to only having 6 dimensions instead of the required 7, as discussed at the beginning of this Section. With this, we see that IIA reductions are just a special case of M-theory reductions. We will consider the more general M-theory reduction now.

Theorem 4.2 (M-Theory Reduction on General G_2-Manifolds)**.** *We begin with the action from Definition 3.2, with the field content being an 11D metric and a 3-form:*

$$S = \frac{1}{2\kappa_{(11)}^2}\int_{M_{11}} \tilde{R}\star 1 - \frac{1}{2}\tilde{F}_4\wedge\star\tilde{F}_4 - \frac{1}{6}\tilde{A}_3\wedge\tilde{F}_4\wedge\tilde{F}_4 \tag{312}$$

where $\tilde{F}_4 = d\tilde{A}_3$. We let $M_{11} = \mathbb{M}_4\times G$ where G is a 'proper' G_2-Manifold, i.e. $hol(g) = G_2$ for our metric on G.

We will follow [11] as a guide here. From Lemma 4.1 we know that the only unique Betti numbers are b_2 and b_3, so we can make make the expansion of our 11D 3-form: [11]

$$\tilde{A}_3(x,z) = A^I(x)\wedge\omega_I(z) + p^k(x)\alpha_k(z) \tag{313}$$

where $\{\omega_I\}_{I=1}^{b_2}$ is a basis of $H^2(G)$, $\{\alpha_k\}_{k=1}^{b_3}$ is a basis of $H^3(G)$, A^I are 1-forms on $\mathbb{M}_4$, and p^k are scalars on $\mathbb{M}_4$.

We also know from [16] that the 11D metric will be expanded into a 4D metric and b_3 scalar fields. This is because we can split the 11D metric into a 4D metric and fill the remaining rows and columns with $S^k\alpha_k$. Thus, similar to Theorem 3.1, we get

$$\tilde{\mathscr{R}}\star_{11} 1 = \mathscr{R}\star_{10}\wedge\star_1 1 + d_4S^k\wedge\alpha_k\wedge\star_{11}d_4S^L\wedge\alpha_L \tag{314}$$

where S^k are b_3 scalar fields on $\mathbb{M}_4$, using notation of these scalars from [11]. We can begin our expansion one term at a time, beginning with the field-strength term of $\tilde{A}_3$:

$$\int_{\mathbb{M}_4\times G}\tilde{F}_4\wedge\star_{11}\tilde{F}_4 =$$

$$\int_{\mathbb{M}_4\times G}\{(d_4A^I\wedge\omega_I + d_4p^k\wedge\alpha_k)\wedge\star_{11}(d_4A^J\wedge\omega_J + d_4p^l\wedge\alpha_l)\} \tag{315}$$

$$= \int_{\mathbb{M}_4\times G}\{d_4A^I\wedge\omega_I\wedge\star_4 d_4A^J\wedge\star_7\omega_J + d_4p^k\wedge\alpha_k\wedge\star_4 d_4p^l\wedge\star_7\alpha_l \tag{316}$$

$$+ \underline{d_4A^I\wedge\omega_I\wedge\star_4 d_4p^l\wedge\star_7\alpha_l} + \underline{d_4p^k\wedge\alpha_k\wedge\star_4 d_4A^J\wedge\star_7\omega_J}\}$$

$$= \int_{\mathbb{M}_4} -G_{IJ}dA^I\wedge\star dA^J - H_{kl}dp^k\wedge\star dp^l \tag{317}$$

where the underlined terms in Equation (316) vanish due to having a 5-form on a 4D space and an 8-form on a 7D space respectively. Here,

$$G_{IJ} = \int_G \omega_I\wedge\star_7\omega_J, \;\; H_{kl} = \int_G \alpha_k\wedge\star_7\alpha_l \tag{318}$$

The Chern-Simons term here is smaller than the one in Theorem 3.1, but we'll still use the same approach as last time here. However, we won't go into much detail - of 8 terms we have 3 survive due to having the correct dimensions. However, two of these cancel each other out due to being odd permutations of one another. Therefore, using our expansion of $\tilde{A}_3$, we get:

$$\int_{\mathbb{M}_4\times G}(A^I\wedge\omega_I+p^k\alpha_k)\wedge(dA^J\wedge\omega_J+dp^l\wedge\alpha_l)\wedge(dA^K\wedge\omega_K+dp^m\wedge\alpha_m)$$
$$=\int_{\mathbb{M}_4}\kappa_{IJk}p^k dA^I\wedge dA^J \tag{319}$$

where $\kappa_{IJk}=\int_G\omega_I\wedge\omega_J\wedge\alpha_k$ are the intersection numbers.

Finally, then, we get our 4D action:

$$S_4=\frac{1}{2\kappa_{(4)}^2}\int_{\mathbb{M}_4}\{\mathscr{R}\star 1+\frac{1}{2}G_{IJ}dA^I\wedge\star dA^J \tag{320}$$
$$+\frac{H_{kl}}{2}(dp^k\wedge\star dp^l+2dS^k\wedge\star dS^l)-\frac{\kappa_{IJk}}{6}p^k dA^I\wedge dA^J\}$$

which thankfully matches the result in [11]. Our field content is then given by:

- *4D Ricci scalar: $\mathscr{R}$*
- *b_2 1-forms: A^I*
- *$2b_3$ scalars: p^k, S^k*

To achieve the correct content of the standard model we require the Betti numbers to match the correct number of each field in the standard model, as well as for these scalars and forms to transform correctly under various symmetries and mechanisms.

Now that we've seen the dimensional reduction of M-theory, we can begin to consider some example manifolds. As mentioned in the previous subsection, having $hol(g)=G_2$ is the assumption we make in our reduction, so the example we gave with this condition will be preferred. We can still see what the example with $hol(g)\subset G_2$ would give us though:

Example 4.3 ($(T^6/\mathbb{Z}_3)\times S^1$ Reduction)**.** *Now that we've seen a reduction on a general G_2-manifold G. e can consider the examples that we constructed. We can use the manifold from Example 4.1 as our manifold G, where we have*

$$b_2=36,\ b_3=38 \tag{321}$$

such that the field content we get is

- *4D Ricci scalar: $\mathscr{R}$*
- *36 1-forms: A^I*
- *76 scalars: p^k, S^k*

However we might not be able to fully trust this reduction, as we haven't taken into account the fact that we also have $b_1 = 1$, which could have appeared in our general reduction at some point.

Example 4.4 (T^7 Orbifold Reduction)**.** *Now let G be the manifold in Example 4.2, which had $hol(g) = G_2$. The unique non-trivial Betti numbers of this orbifold were*

$$b_2 = 12,\ b_3 = 7 \tag{322}$$

so our resulting field content is then given by:

- *4D Ricci scalar: $\mathscr{R}$*
- *12 1-forms: A^I*
- *14 scalars: p^k, S^k*

As discussed, the fact that this manifold has strictly G_2 holonomy means it is more true to our assumptions in the reduction. What's interesting about this example is that the standard model has 12 gauge bosons, which can be written as 1-forms. Therefore, the reduction on this manifold gives us the correct number of 1-forms that we would expect to see in the standard model. The number of scalar fields is still too low however.

Now that we've seen a very promising potential unification of gravity with the standard model, we would like to check that this reduction is consistent. When we reduced IIA, we did not check for consistency as it did not resemble the standard model and so we were not worried about whether it could be perceived as a reliable physical theory. We will show the consistency of M-theory reductions now:

Theorem 4.3 (Consistency of M-Theory Reduction)**.** *As mentioned after Theorem 3.1, we need to check that the 11D equations of motion and the 4D equations of motion are consistent. We can do this by finding the equations of motion in 11D and 4D respectively, and then showing that substituting the 4D equations into the 11D satisfy the 11D equation.*

For the higher-dimensional action, given in Definition 3.2, we can vary in $\tilde{A}_3$:

$$\begin{aligned}\delta_A S = \int \{ & -\frac{1}{2} d(\delta\tilde{A}_3) \wedge \star d\tilde{A}_3 - \frac{1}{2} d\tilde{A}_3 \wedge \star d(\delta\tilde{A}_3) \qquad (323)\\ & -\frac{1}{6}(\delta\tilde{A}_3 \wedge d\tilde{A}_3 \wedge d\tilde{A}_3 + \tilde{A}_3 \wedge d(\delta\tilde{A}_3) \wedge d\tilde{A}_3 + \tilde{A}_3 \wedge d\tilde{A}_3 \wedge d(\delta\tilde{A}_3))\} \\ = \int \{ & \delta\tilde{A}_3 \wedge d \star d\tilde{A}_3 + \frac{1}{6}\delta\tilde{A}_3 \wedge d\tilde{A}_3 \wedge d\tilde{A}_3\} \qquad (324)\\ = \int & \delta\tilde{A}_3 \wedge (d \star d\tilde{A}_3 + \frac{1}{6} d\tilde{A}_3 \wedge d\tilde{A}_3) \qquad (325)\end{aligned}$$

$$\Rightarrow d \star d\tilde{A}_3 + \frac{1}{6} d\tilde{A}_3 \wedge d\tilde{A}_3 = 0 \tag{326}$$

where we have obtained the equation of motion of $\tilde{A}_3$ in Equation (326). In Equation (324) we used Theorem 1.4 to rearrange the derivatives on the first two terms of the previous line, and saw that the final two terms of Equation (326) cancel each other our by permuting the wedge products. While strange at first, we will see that writing the equation of motion in the following form will assist us in proving the consistency of our reduction:

$$[d\star d\tilde{A}_3 + \frac{1}{6}d\tilde{A}_3 \wedge d\tilde{A}_3] \wedge \omega_K \wedge \alpha_m = 0 \tag{327}$$

For the reduction in Theorem 4.2 to be consistent, we require that the equations of motions obtained from the 4D action satisfy this equation of motion when the expansion is made. To do this, we consider the equations of motion for A^I, p^k in the 4D action by first varying A^I:

$$\delta_A S_4 = \int -G_{IJ} d(\delta A^I) \wedge \star dA^J - \frac{\kappa_{IJk}}{3} p^k d(\delta A^I) \wedge dA^J \tag{328}$$

$$= \int G_{IJ} \delta A^I \wedge d\star dA^J + \frac{\kappa_{IJk}}{3} dp^k \wedge \delta A^I \wedge dA^J \tag{329}$$

$$= \int \delta A^I \wedge (G_{IJ} d\star dA^J - \frac{\kappa_{IJk}}{3} dp^k \wedge dA^J) \tag{330}$$

$$\Rightarrow G_{IJ} d\star dA^J - \frac{\kappa_{IJk}}{3} dp^k \wedge dA^J = 0 \tag{331}$$

where Equation (331) gives the first equation[8] *of motion in A^I, p^k. Again, to write this equation in a peculiar way to assist us later:*

$$\omega_I \wedge \star_7 \omega_J \wedge (d\star dA^J) - \frac{1}{3}\omega_I \wedge \omega_J \wedge \alpha_k \wedge (dp^k \wedge dA^J) = 0 \tag{332}$$

We can now vary the action by p^k:

$$\delta_p S_4 = \int H_{kl} d(\delta p^k) \wedge \star dp^l - \frac{\kappa_{IJk}}{6} \delta p^k dA^I \wedge dA^J \tag{333}$$

$$= \int -\delta p^k \wedge (H_{kl} d\star dp^l + \frac{\kappa_{IJk}}{6} dA^I \wedge dA^J) \tag{334}$$

$$\Rightarrow H_{kl} d\star dp^l + \frac{\kappa_{IJk}}{6} dA^I \wedge dA^J = 0 \tag{335}$$

where Equation (335) gives us the second equation of motion. A final strange rewriting of these equations of motion is required:

$$\alpha_k \wedge \star_7 \alpha_l \wedge (d\star dp^l) + \frac{1}{6}\omega_I \wedge \omega_J \wedge \alpha_k \wedge (dA^I \wedge dA^J) = 0 \tag{336}$$

What we want to do now is to substitute our expansion of $\tilde{A}_3$ from Equation (313) into Equation (327) and then use Equations (332) and (336) to show that the 11D equations of motion are

[8] Technically there are b_2 equations of motion here

satisfied:

$$\begin{aligned}
&[d\star d(A^I\wedge\omega_I+p^k\alpha_k)+\frac{1}{6}d(A^I\wedge\omega_I+p^k\alpha_k)\wedge d(A^J\wedge\omega_J+p^l\wedge\alpha_l)]\wedge\omega_K\wedge\alpha_m\\
&=[(d\star_4 dA^I)\wedge\star_7\omega_I-(d\star_4 dp^k)\wedge\star_7\alpha_k+\frac{1}{6}dA^I\wedge dA^J\wedge\omega_I\wedge\omega_J\\
&+\frac{1}{3}dA^I\wedge dp^l\wedge\omega_I\wedge\alpha_l-\frac{1}{6}dp^k\wedge dp^l\wedge\alpha_k\wedge\alpha_l]\wedge\omega_K\wedge\alpha_m
\end{aligned}\tag{337}$$

$$\begin{aligned}
&=(d\star_4 dA^I)\wedge\star_7\omega_I\wedge\omega_K\wedge\alpha_m-(d\star_4 dp^k)\wedge\star_7\alpha_k\wedge\alpha_m\wedge\omega_K\\
&-\frac{1}{6}dA^I\wedge dA^J\wedge\omega_I\wedge\omega_J\wedge\alpha_m\wedge\omega_K-\frac{1}{3}dp^k\wedge dA^I\wedge\omega_I\wedge\omega_K\wedge\alpha_k\wedge\alpha_m\\
&-\frac{1}{6}dp^k\wedge dp^l\wedge\alpha_k\wedge\alpha_l\wedge\alpha_m\wedge\omega_K
\end{aligned}\tag{338}$$

$$\begin{aligned}
&=[\omega_K\wedge\star_7\omega_I\wedge(d\star_4 dA^I)-\frac{1}{3}\omega_I\wedge\omega_K\wedge\alpha_k\wedge dp^k\wedge dA^I]\wedge\alpha_m\\
&-[\alpha_m\wedge\star_7\alpha_k\wedge(d\star_4 dp^k)+\frac{1}{6}\omega_I\wedge\omega_J\wedge\alpha_m\wedge(dA^I\wedge dA^J)]\wedge\omega_K=0
\end{aligned}\tag{339}$$

where we see that the equations of motion (332) and (336) in 4D now satisfy the 11D equation of motion. In the final term of Equation (338) we used the fact that the wedge of the three 3-forms will vanish due to being a 9-form of a 7D manifold. Therefore, the action for A^I, p^k is consistent. We will not consider the consistency of the Ricci scalar reduction as it would be longer and more complicated than what we have just done.

Theorem 4.4 (Gauge Symmetry of M-theory Reduction)**.** *The gauge symmetry obtained from the reduction of M-theory on a G_2-manifold contains the gauge symmetry of the standard model:*

$$SU(3)\times SU(2)\times U(1)\subset G_2\times G_2 \tag{340}$$

where $G_2\times G_2$ is the resulting gauge symmetry we obtain from our reduction in Theorem 4.2.

Proof: *We have our 11D manifold as $\mathbb{M}_4\times G$, where G is a non-abelian Lie group G_2. [22] says that in the case of a reduction on $\mathbb{M}_4\times G$, where G is a non-abelian lie group, the resulting gauge symmetry group is $G\times G$. So, in our case we have a resulting $G_2\times G_2$ symmetry group. From [10] we know that*

$$SU(3)\subset G_2 \tag{341}$$

which means that our resulting gauge symmetry has

$$SU(3)\times SU(3)\subset G_2\times G_2 \tag{342}$$

From [1] we know that

$$SU(2)\times U(1)\subset SU(3) \tag{343}$$

which means finally that our gauge symmetry group $G_2\times G_2$ contains the gauge symmetry group of the standard model:

$$SU(3)\times SU(2)\times U(1)\subset G_2\times G_2 \quad \square \tag{344}$$

This section has shown a very promising attempt to unify gravity with the standard model: We have a gauge symmetry containing the standard model's gauge symmetry group, as well as predicting scalar fields and 1-forms (which can be viewed as vector bosons), all while having a term in the action corresponding to the Einstein-Hilbert action in 4D. Our only remaining issue is that the exact number of scalar fields and 1-forms that have been predicted depends on the chosen G_2-manifold, of which there could potentially be infinitely many to choose from[9]. There also exists the issue of not knowing what the fields in our reduction correspond to in the standard model, as the field content in our 4D theory is bosonic, whereas the standard model contains both bosonic and fermionic fields. We will address this in our conclusions.

[9] The number of existing G_2-manifolds is unknown.

5 Conclusions

We began this thesis by considering an important motivating example for unified theories, namely Kaluza-Klein theory. This theory gave us a unification of gravity and electromagnetism from just one extra dimension, and so we set out to consider theories of even greater dimension that could give us something close to a unification of gravity with the standard model. Before jumping right into physical theories, we delved into some geometry that could help us describe more interesting extra dimensions. The required extra dimensions for a 10D string theory, Calabi-Yau manifolds, were covered in great depth and we gave some insightful examples with a particular focus on orbifold constructions. An important aspect of our examples were the Betti numbers or Hodge numbers of the manifold. One of the most interesting parts of this thesis has been realising that these seemingly simple topological numbers end up playing a crucial role in the dimensional reductions of our theories - they tell us how many fields our 4D theory will have. When we reduced Type IIA string theory we obtained a 4D theory with a diverse and plentiful array of fields, but this didn't really look like the standard model.

Here, we chose to move to M-theory reductions, as this is the minimal dimension that we can have the gauge symmetry of the standard model, as well as being the maximum dimension possible for supergravity theories [22]. First we discussed the required extra 7 dimensional manifolds for M-theory, G_2-manifolds. We gave more orbifold constructions and found their Betti numbers, and then used these as our extra dimensions for M-theory. We performed the dimensional reduction of M-theory and saw something incredible - we obtain a theory in 4D that has a number of scalar fields and vector fields (1-forms) which contain the standard model's gauge symmetry group as a subgroup of their gauge symmetry. This was personally a great surprise for the author, who did not expect to see such a promising result from the dimensional reduction.

Something to take into consideration is the fact that there is a considerably large and potentially infinite number of manifolds to represent the extra dimensions of our string theory and M-theory reductions. The standard model will have a given number of scalar and vector fields and so we need to choose the Hodge numbers or Betti numbers to match this number. This leads us to The Swampland. Every distinct manifold that we choose to represent the extra dimensions of our theory will produce a different 4D theory because of the different Betti numbers or Hodge numbers of the manifold, most of which will be 'false vacua', i.e. theories that don't describe the universe we live in [21].

While the results seem incredibly promising as a potential unification of gravity with the standard model, it must be pointed out that these fields come from the bosonic section of the supergravity action in 11D. This means that the action is supersymmetric [17], while the standard model is not. Thus, we are not quite at our final unified theory yet. It occurs to the author that an interesting future area of study could be supersymmetric extensions of the standard model in an attempt to try and connect it to the dimensional reduction of M-theory. Alternatively, perhaps this route ceases to be anything more than a pleasing mathematical coincidence without a way of connecting it to a realistic physical theory.

A Homology

For our discussion of Homology, we shall follow the arguments of [5] more so than [14]. This is because [14] approaches the topic from the viewpoint of simplicial complexes, whereas [5] takes a sub-manifold approach which aligns more with this project. The reason I have chosen to discuss homology is because of it's relationship to cohomology, and helps us understand the physical meaning of cohomology for a manifold.

Before we describe what Homology is, we need to introduce the concept of chains and cycles. We borrow these definitions directly from [5].

Definition A.1 (Chains)**.** *A p-chain c_p of manifold M is defined as*

$$c_p = \Sigma a_i N_i \tag{345}$$

where a_i is just a constant (can be integer, real, complex) and N_i is a p-dimensional oriented submanifold of M. [5]

This is then just a collection of submanifolds with a constant assigned to each. To define a cycle, we need to remember that ∂N means the boundary of a manifold N, and that $\partial^2 N = 0$, i.e. the boundary of a boundary is empty. A good example of this fact is that the boundary of a disc is a circle, and a circle has no boundary [5].

Note that ∂ is linear, so if we try to take the boundary of a p-chain c_p, we can just write it as the sum of the boundaries of the submanifolds:

$$\partial c_p = \Sigma a_i \partial N_i \tag{346}$$

where ∂c_p is thus a (p-1)-chain [5] (think about the example of a disc: a disc is a 2-chain, and it's boundary is a circle, a 1-chain, so the boundary operator takes us down a dimension.)

Definition A.2 (Cycles)**.** *A p-cycle is a p-chain c_p such that $\partial c_p = 0$ [5]. That is, a p-chain where every submanifold N_i in the chain has no boundary.*

Now something we would like to consider are cycles that are also not boundaries. Obviously we can have a chain with $\partial c_p = 0$ by simply choosing that each $N_i = \partial M_i$ i.e. that each submanifold is the boundary of some other submanifold. This is similar to how every exact differential form is closed. This would give us:

$$\partial c_p = \Sigma a_i \partial\partial M_i = 0 \tag{347}$$

but this is a rather trivial case.

The Homology Group is what we would like to consider - the set of cycles that are without boundary but are not boundaries.

Definition A.3 (Homology)**.** *Let $\mathbb{C}_p = \{c_p \mid c_p$ is a p-cycle on M$\}$ be the set of p-cycles on M and $\mathbb{B}_p = \{b_p | b_p = \partial a_{p+1}\}$ be the set of p-chains that are boundaries of a (p+1)-chain such that for*

$b_p \in \mathbb{B}_p$ we have $\partial b_p = \partial\partial a_{p+1} = 0$, i.e. b_p is a trivial p-cycle. We then define the Homology of M to be

$$\mathbb{H}_p = \mathbb{C}_p/\mathbb{B}_p \tag{348}$$

which is just the quotient of the two aforementioned sets [5]. This gives us the set of non-trivial p-cycles. For any two cycles that differ by a boundary, we identify them, i.e. [5]

$$c_p \sim c_p + \partial a_{p+1} \tag{349}$$

We also have an important relationship between homology and de Rham cohomology, given by the following theorem:

Theorem A.1 (Equivalence of homology and de Rham cohomology)**.** *Let $\mathbb{H}_p$ be the homology of M and $\mathbb{H}^p$ be the de Rham cohomology of M. Then we have that these spaces are dual to each other and are thus isomorphic [5]:*

$$\mathbb{H}_p \cong \mathbb{H}^p \tag{350}$$

Corollary A.1. *Let us have a manifold M with a given homology and cohomology, $\mathbb{H}_p$ and $\mathbb{H}^p$. Then due to Theorem A.1 we have that*

$$dim\mathbb{H}_p = dim\mathbb{H}^p \tag{351}$$

This tells us that a harmonic form corresponds to a submanifold of M without boundary that is not itself a boundary.

B Additional Kaluza-Klein Examples

As well as the original Kaluza-Klein example of unifying gravity and electromagnetism, we can use the same concept for other theories. Going from a simpler theory in a higher dimension and eliminating the extra dimension via integration and Fourier expansions can give us a more complicated theory in a lower dimension. The following examples both show this. We can follow [18] for our inspiration for this Appendix.

Example B.1 (Klein-Gordon Equation)**.** *Let us pick a Lagrangian that gives us the Klein-Gordon equations of motion*

$$\mathscr{L} = \partial_I\bar{\phi}\partial^I\phi + m^2|\phi|^2 \tag{352}$$

where $I \in \{0,..,3,4\}$ such that $I = 4$ corresponds to a coordinate on S^1. That is, we have action

$$S = \int_{\mathbb{M}_4\times S^1} d^4xdy[\partial_I\bar{\phi}\partial^I\phi + m^2|\phi|^2] \tag{353}$$

where y is the coordinate on S^1. What we can do is split the Lagrangian up into a part over $\mathbb{M}_4$ and a part over S^1 and then use a Fourier expansion like in Definition 1.2:

$$S = \int_{\mathbb{M}_4\times S^1} d^4xdy[\partial_\mu\bar{\phi}\partial^\mu\phi + \partial_4\bar{\phi}\partial^4\phi + m^2|\phi|^2] \tag{354}$$

$$\phi(x,y) = \sum_{n=-\infty}^{\infty} \phi_n(x) e^{iny/R} \tag{355}$$

where R is the radius of the S^1. Substituting Equation (355) into (354) gives us:

$$\begin{aligned} S = \int_{\mathbb{M}_4 \times S^1} d^4x dy[\partial_\mu (\sum_n \bar{\phi}_n e^{-iny/R})(\partial^\mu \sum_m \phi_m e^{imy/R}) \\ + \partial_4 (\sum_n \bar{\phi}_n e^{-iny/R})(\partial^4 \sum_m \phi_m e^{imy/R}) \\ + m^2 (\sum_n \bar{\phi}_n e^{-iny/R})(\sum_m \phi_m e^{imy/R})] \end{aligned} \tag{356}$$

$$\begin{aligned} = \int_{\mathbb{M}_4 \times S^1} d^4x dy[(\sum_n (\partial_\mu \bar{\phi}_n) e^{-iny/R})(\sum_m (\partial^\mu \phi_m) e^{imy/R}) \\ + (\sum_n \bar{\phi}_n (\partial_4 e^{-iny/R}))(\sum_m \phi_m (\partial^4 e^{imy/R})) \\ + m^2 (\sum_{n,m} \bar{\phi}_n \phi_m e^{-iny/R} e^{imy/R})] \end{aligned} \tag{357}$$

$$\begin{aligned} = \int_{\mathbb{M}_4} d^4x \int_0^{2\pi R} e^{i(m-n)y/R} dy[(\sum_{n,m} (\partial_\mu \bar{\phi}_n)(\partial^\mu \phi_m)) \\ + (\sum_{n,m} -\frac{in}{R} \bar{\phi}_n (\frac{im}{R} \phi_m)) \\ + m^2 (\sum_{n,m} \bar{\phi}_n \phi_m)] \end{aligned} \tag{358}$$

$$\begin{aligned} = 2\pi R \int_{\mathbb{M}_4} d^4x[(\sum_n (\partial_\mu \bar{\phi}_n)(\partial^\mu \phi_n)) \\ + (\sum_n \frac{n^2}{R^2} |\phi_n|^2) \\ + m^2 (\sum_n |\phi_n|^2)] \end{aligned} \tag{359}$$

where we used the identity $\int_0^a e^{i(n-m)x} dx = a\delta^{nm}$ to get from Equation (358) to (359). So finally we end up with:

$$S = \int_{\mathbb{M}_4} d^4x[\partial_\mu \bar{\phi}_0 \partial^\mu \phi_0 + m^2 |\phi_0|^2 + \sum_{n \neq 0} (\partial_\mu \bar{\phi}_n \partial^\mu \phi_n + (\frac{n^2}{R^2} + m^2)|\phi_n|^2)] \tag{360}$$

which is a massive complex scalar field of mass m^2 and a tower of massive complex scalar fields of mass $\frac{n^2}{R^2} + m^2$. The fact that n is an integer means that from using this Kaluza-Klein compactification we have obtained a massive scalar field and a tower of massive scalar fields of quantized mass.

So from this example we've seen that Kaluza-Klein theory not only helps us unify forces as in Subsection 1.1, but also enables us to achieve a quantized theory from a classical theory in higher dimensions. Not only can we do this for a scalar field, but we can do this for a vector field as well, which we'll see in the next example.

Example B.2 (Compactifying vector fields)**.** *Now we let our Lagrangian be*

$$\mathscr{L} = -\frac{1}{4}F_{IJ}F^{IJ} \tag{361}$$

where $F_{IJ} = \partial_I A_J - \partial_J A_I$, again with $I,J \in \{0,...3,4\}$ where the fourth coordinate is over S^1. This is just the Lagrangian of a massless vector field. Therefore our action is defined as

$$S = \int_{\mathbb{M}_4 \times S^1} d^4xdy[-\frac{1}{4}F_{IJ}F^{IJ}] \tag{362}$$

$$= \int_{\mathbb{M}_4 \times S^1} d^4xdy[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}F_{\mu 4}F^{\mu 4}] \tag{363}$$

where the second term has been multiplied by a factor of two as there are two copies of $F_{\mu 4}F^{\mu 4}$ due to symmetry. The next step is to use the expansion

$$A_I = \sum_{n=-\infty}^{\infty} A_I^{(n)} e^{iny/R} \tag{364}$$

with the same expansion for A^I. Additionally, as a vector field can be seen as just a collection of scalar fields, we can relabel $A_5 = A^5 = \phi$.

We first need to expand the terms into terms of A_I before substituting in these expansions, so the first term becomes

$$F_{\mu\nu}F^{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \tag{365}$$

$$= 2\partial_\mu A_\nu \partial^\mu A^\nu - 2\partial_\mu A_\nu \partial^\nu A^\mu \tag{366}$$

$$= 2\partial_\mu (\sum_n A_\nu^{(n)} e^{iny/R})\partial^\mu (\sum_m A_{(m)}^\nu e^{imy/R}) - 2\partial_\mu (\sum_n A_\nu^{(n)} e^{iny/R})\partial^\nu (\sum_m A_{(m)}^\mu e^{imy/R}) \tag{367}$$

$$= e^{i(n+m)y/R} \sum_{n,m} [2\partial_\mu A_\nu^{(n)} \partial^\mu A_{(m)}^\nu - 2\partial_\mu A_\nu^{(n)} \partial^\nu A_{(m)}^\mu] \tag{368}$$

$$= e^{i(n+m)y/R} \sum_{n,m} F_{\mu\nu}^{(n)} F_{(m)}^{\mu\nu} \tag{369}$$

and the second term becomes

$$F_{\mu 4}F^{\mu 4} = (\partial_\mu A_4 - \partial_4 A_\mu)(\partial^\mu A^4 - \partial^4 A^\mu) \tag{370}$$
$$= \partial_\mu \phi \partial^\mu \phi + \partial_4 A_\mu \partial^4 A^\mu - \partial_4 A_\mu \partial^\mu \phi - \partial_\mu \phi \partial^4 A^\mu \tag{371}$$
$$= \partial_\mu \sum_n (\phi_n e^{iny/R}) \partial^\mu \sum_m (\phi_m e^{imy/R}) + \partial_4 \sum_n (A_\mu^{(n)} e^{iny/R}) \partial^4 \sum_m (A^\mu_{(m)} e^{imy/R}) \tag{372}$$
$$- \partial_4 \sum_n (A_\mu^{(n)} e^{iny/R}) \partial^\mu \sum_m (\phi_m e^{imy/R}) - \partial_\mu \sum_n (\phi_n e^{iny/R}) \partial^4 \sum_m (A^\mu_{(m)} e^{imy/R})$$
$$= e^{i(n+m)y/R} \sum_{n,m} [\partial_\mu \phi_n \partial^\mu \phi_m + \frac{i^2 nm}{R^2} A_\mu^{(n)} A^\mu_{(m)} - \frac{in}{R} A_\mu^{(n)} \partial^\mu \phi_m - \frac{im}{R} A^\mu_{(m)} \partial_\mu \phi_n] \tag{373}$$

So if we then integrate the exponential terms over the S^1, we get that $m = -n$ and we multiply by $2\pi R$, which gives the effective 4D action as

$$S = 2\pi R \int_{\mathbb{M}_4} d^4x - \frac{1}{2} \sum_n [\frac{1}{2} F_{\mu\nu}^{(n)} F^{\mu\nu}_{(-n)} + \partial_\mu \phi_n \partial^\mu \phi_{(-n)} \tag{374}$$
$$+ \frac{n^2}{R^2} A_\mu^{(n)} A^\mu_{(-n)} + \frac{in}{R} (A^\mu_{(-n)} \partial_\mu \phi_n - A_\mu^{(n)} \partial^\mu \phi_{(-n)})]$$

which we can see gives us a massless vector field and a massless scalar field (by letting n=0) as well as a tower of massive vector fields with mass $\frac{n^2}{R^2}$ and some couplings between the vector and scalar fields. To remove this term, we can fix the gauge

$$A_\mu \to A_\mu + \partial_\mu \alpha \tag{375}$$

by letting

$$\alpha = -\frac{iR}{n} \phi_n = \frac{iR}{n} \phi_{(-n)} \tag{376}$$

which results in the action

$$S = 2\pi R \int_{\mathbb{M}_4} d^4x - \frac{1}{2} \sum_n [\frac{1}{2} F_{\mu\nu}^{(n)} F^{\mu\nu}_{(-n)} + \partial_\mu \phi_0 \partial^\mu \phi_0 + \frac{n^2}{R^2} A_\mu^{(n)} A^\mu_{(-n)}] \tag{377}$$

We won't show that this gauge fixing removes the additional term, as it's a rather lengthy and tedious calculation. Therefore, by fixing this gauge we began with a 5D massless vector field and have obtained a massless vector field, a massless scalar field, and a tower of massive vector fields.

It is worth noting that while these are some fun toy examples of what can be done with Kaluza-Klein theory, they are not consistent reductions [18]. Therefore they cannot be considered as realistic physical theories, but are just interesting mathematical results.

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